# Exact Renormalization Group 

## AJ

December 7, 2017

## Contents

1 Introduction ..... 3
2 Classical renormalization: screening ..... 3
2.1 Light of a lamp ..... 3
2.2 Charge in electrolyte ..... 5
2.3 Screening by dipoles ..... 6
2.4 Dynamical screening ..... 7
2.5 Multi-body interactions ..... 7
3 Master formula for self-interacting systems: path integral ..... 8
3.1 Path integral in statistical physics ..... 8
3.2 Path integral in quantum field theories ..... 9
3.3 Generator functional ..... 11
$3.4 \quad \Phi^{4}$ model ..... 12
4 Perturbation theory ..... 13
4.1 Free theories ..... 13
4.2 Interactions ..... 15
5 Functional methods ..... 17
5.1 Free energy ..... 17
5.2 Effective action ..... 18
6 Giving sense the path integral: renormalization ..... 22
6.1 Renormalized perturbation theory ..... 23
7 Changing the regularization ..... 26
7.1 Convergence of the perturbation theory ..... 26
7.2 The regularization dependent action ..... 27
7.3 Exact evolution equation for $\Gamma_{>}$ ..... 29
7.4 Technical addon: the second derivative of the FRG equation ..... 31
7.5 Interpretation of the FRG equation ..... 31
7.6 Characterization of the effective action ..... 32
7.7 An alternative form of the exact RG evolution equation ..... 34
7.8 Renormalization equation in renormalized perturbation theory ..... 35
7.9 LPA and LPA' approximation ..... 36
7.9.1 The evolution of the potential ..... 36
7.9.2 LPA approximation ..... 37
7.9.3 LPA' approximation ..... 38
7.9.4 Evolution of the kernel ..... 39
8 FRG in the scalar model ..... 41
$8.1 \quad \Phi^{4}$ model, non rescaled form ..... 41
8.1.1 Symmetric phase ..... 43
8.1.2 Broken symmetry case ..... 43
8.1.3 The physical values. ..... 44
8.2 The rescaled equation ..... 45
8.3 Fixed point ..... 46
8.4 Effect of higher powers in the potential ..... 47
9 General properties of the exact RG equation ..... 48
9.1 Fixed points, relevant and irrelevant operators ..... 48
9.2 Gaussian fixed points and expansion around it ..... 49
9.3 Dimensional transmutation ..... 50
A Mathematical details ..... 52
A. 1 Surface factors and basic integrals ..... 52
A. 2 Integrals ..... 53
A. 3 Fierz identities ..... 54
A. 4 Symmetric Fourier transformation ..... 55

## 1 Introduction

Renormalization is usually associated to quantum field theory and statistical physics, and as such is considered as something complicated and weird. But it is a much more general notion, and it plays a central role in understanding the physics in general.

Renormalization is in fact a scientific name of the general phenomenon of screening. Screening appears everywhere, we can use it for example to reduce external effects, like we use shading to reduce sunlight, or use shielding to protect our electric devices against external electric fields. The screening can be automatic, for example when the light of a reflector is damped by the dust in the air.

The phenomenon becomes more complicated when the screeners are charges themselves, ie. when there is self-screening. If we ask for example that what is the electric potential of a charge of an ion in an electrolyte, we have to take into the screening effect of other ions. But their potential is also screened, so screening becomes an implicit problem, where screening depends on screened component. This is not only more complicated, but new phenomena can appear: the original interactions between ions become negligible, and new interactions may show up in the system.

In the first section we review the screening as a classical phenomenon, where all of these problems already appear. Then we turn to the more conventional description of renormalization using the formalism of path integral. As we will see, path integral cannot be defined in the whole world simultaneously, therefore it always contains a regularization. Regularization is, however, something that is not fixed by the theory, it is just a way how we treat it, so it must not influence the real physical observables. This invariance property will lead us to the description of the selfconsistent theory of renormalization.

In this way the renormalization can be treated independently on whether we speak about statistical physics, many body theories or quantum field theory. Still the problems show up in these fields of physics are different, and we also want to understand how to deal with renormalization around fixed points, around a massive IR fixed point, how to treat the UV divergences of quantum field theory.

In the last part of the note we speculate on the consequences that renormalization leads to appearance of new phenomena. As we will see, this automatically makes the description of the whole world with a single theory impossible - so the Theory of Everything, albeit it can exist, but has no relevance whatsoever on the theories we already know. It explains why there are so many scientific disciplines, why are they different and why does not use a chemists the Standrad Model to describe molecules.

In human thinking we reveal new phenomena using our intelligence, new species are born in nature through evolution, while new phenomena appear automatically in the course of renormalization process. This suggests that there is a connection between intelligence, evolution and renormalization - and this also means that if we want to build artificial intelligence, the theory of renormalization can be of big help. This connection is discussed in the last section of this note.

## 2 Classical renormalization: screening

As we mentioned in the introduction, renormalization is essentially the same as screening. In this section we examine some simple models in classical systems where screening plays a role.

### 2.1 Light of a lamp

Let us consider a lamp in the night observed from a distance. The brightness of the lamp is usually described by the power radiated out, called luminosty. The observed brightness is characterized by the energy current density $\boldsymbol{F}$; its meaning is that the light power going through a directed $d \boldsymbol{A}$ surface is $d P=\boldsymbol{F} d \boldsymbol{A}$ (its analogous notion in electrodynamics is the Poynting vector). The luminosity is connected to $\boldsymbol{F}$ by $L=\oint_{F} d \boldsymbol{A F}$.

If the lamp was a point source, then the observed brightness (luminosity) of the lamp would be uniformly distributed on a sphere. Then $\boldsymbol{F} \| \boldsymbol{e}_{r}$ directed outwards, and so $L=4 \pi r^{2} F$, where $r$ is the distance from the source. In a realistic case, for example if we use a reflector mirror, but the light source can be considered pointlike, the original brightness is radiated out with some angle distribution, so $\boldsymbol{F}=F(r, \Omega) \boldsymbol{e}_{r}$. Let us consider a small volume with sides parallel to $\boldsymbol{e}_{r}$, its with shall be $d r$. The power straming into this surface
is $F(r, \Omega) r^{2} d \Omega$, the power coming out is $F(r+d r, \Omega)(r+d r)^{2} d \Omega$. Energy conservation requires

$$
\begin{equation*}
F(r+d r, \Omega)(r+d r)^{2} d \Omega=F(r, \Omega) r^{2} d \Omega \tag{1}
\end{equation*}
$$

This yields differential equations independently for all $\Omega$ solid angles.

$$
\begin{equation*}
\frac{\partial F(r, \Omega)}{\partial r}=-\frac{2}{r} F(r, \Omega) \tag{2}
\end{equation*}
$$

with the solution

$$
\begin{equation*}
F(r, \Omega)=\frac{\alpha_{0}(\Omega)}{r^{2}} \tag{3}
\end{equation*}
$$

Energy conservation therefore requires a $r^{-2}$ decrease in this function. The luminosity of the light source is $L=r^{2} \int d \Omega F(r, \Omega)=\int d \Omega \alpha_{0}(\Omega)$, so $\alpha_{0}$ is proportional to the absolute luminosity of the light source. By formal analogy later it is also called "coupling constant", or "charge" of the source. For simplicity later we will omit the angle dependence of $\alpha_{0}$, therefore in this spherically symmetric case

$$
\begin{equation*}
F(r)=\frac{\alpha_{0}}{r^{2}} \tag{4}
\end{equation*}
$$

This ideal case rarely happens. Usually there are some disturbances in the air, dust or insects that absorb some part of the original radiation. Let us assume that the density of the disturbance is $n$, and the cross section of each absorbing particle is $\sigma$. If we write up the energy balance in the same volume we considered before, we can say that the power straming into this surface is again $F(r) r^{2} d \Omega$, the power coming out is $F(r+d r)(r+d r)^{2} d \Omega$, but there is also energy loss in this case. The number of particles absorbing energy is $d N=n d r r^{2} d \Omega$, altogether they cover $d A=\sigma d N$ surface (assuming that they do not overlap in this narrow volume). The surface where the energy flows into this volume is therefore smaller than the original $r^{2} d \Omega$ by $d A$. So energy conservation reads:

$$
\begin{equation*}
F(r+d r)(r+d r)^{2} d \Omega=F(r)\left(r^{2} d \Omega-\sigma n d r r^{2} d \Omega\right) \tag{5}
\end{equation*}
$$

This leads to a modified differential equation

$$
\begin{equation*}
\frac{d F}{d r}=-\left(\frac{2}{r}+\sigma n\right) F \tag{6}
\end{equation*}
$$

This equation can be solved for any $n(r)$ function

$$
\begin{equation*}
F(r)=F\left(r_{0}\right) \frac{r_{0}^{2} e^{-\int_{r_{0}}^{r} d r^{\prime} \sigma n\left(r^{\prime}\right)}}{r^{2}} \tag{7}
\end{equation*}
$$

If $n$ happens to be constant, then this means

$$
\begin{equation*}
F(r)=F\left(r_{0}\right) \frac{r_{0}^{2} e^{-n \sigma\left(r-r_{0}\right)}}{r^{2}} . \tag{8}
\end{equation*}
$$

We can interpret this result differently. One is the above interpretation, namely that we know that there is a constant light source, and the medium is active, so we must use a different formula to compute the distance dependence of the observed luminostiy. Another view is when we do not know about the dust, or we do not want to know about it, and interpret the above formula that the luminosity of the light source depends on the distance where we observe its light. Indeed, if we use

$$
\begin{equation*}
\alpha(r)=r_{0}^{2} F\left(r_{0}\right) e^{-n \sigma\left(r-r_{0}\right)}=\alpha_{0} e^{-n \sigma\left(r-r_{0}\right)} \tag{9}
\end{equation*}
$$

then the observed luminosity reads

$$
\begin{equation*}
F(r)=\frac{\alpha(r)}{r^{2}} \tag{10}
\end{equation*}
$$

According to this formula the strength of the light source is distance dependent, we can call it "running" with distance.

Although at the first glance this interpretation seems to be somewhat artificial, it is in fact a very cosy approach. If we do not know about the screening mechanisms then the best we can do is to use a description where we forget about the medium, but we incorporate its effects as distance dependent physical "constants". In the above example we know that the power radiated out by the lamp is constant, so in this system it is a "physical constant". But due to the medium effects we reinterpret - renormalize - this physical constant as a distance-dependent "running" quantity. This is the mechanism we follow in all systems where we can not take into account the details of the medium. This is the physical origin of renormalization.

### 2.2 Charge in electrolyte

Let us have some more examples of screened systems. In electrodynamics, if we have moving charges (like in electrolytes), the total charge of the system is $\varrho_{\text {ext }}+\varrho$ where $\varrho_{\text {ext }}$ is the external charge put in the system, $\varrho$ is the charge coming from the moving ions of the electrolyte. For simplicity we will assume that there is only one species of ions that can move in the electrolyte, its density is denoted by $n$, its charge is $q$. Then $\varrho=q n$, and the first Maxwell equation reads

$$
\begin{equation*}
\operatorname{div} \boldsymbol{E}=\frac{1}{\varepsilon_{0}}\left(\varrho_{e x t}+q n\right) \tag{11}
\end{equation*}
$$

The conservation of charge requires

$$
\begin{equation*}
\partial_{t}(q n)+\operatorname{div} \boldsymbol{j}=0 \tag{12}
\end{equation*}
$$

where $\boldsymbol{j}$ is the total electric current. It has two sources: one is the electric field which tries to accelerate the charges. In the conducting regime we have a conduction $\sigma$, then

$$
\begin{equation*}
\boldsymbol{j}_{\text {electric }}=\sigma \boldsymbol{E} \tag{13}
\end{equation*}
$$

The other source is the diffusion of the particles. Usually one assumes that the diffusion current is proportional to the gradient of the particle density. The electric current stemming from this effect reads

$$
\begin{equation*}
\boldsymbol{j}_{d i f f}=-q D \nabla n \tag{14}
\end{equation*}
$$

So we have

$$
\begin{equation*}
\partial_{t}(q n)+\operatorname{div}(\sigma \boldsymbol{E}-q D \boldsymbol{\nabla} n)=0 \tag{15}
\end{equation*}
$$

using the Maxwell equation this reads

$$
\begin{equation*}
\partial_{t} n+\frac{\sigma}{q \varepsilon_{0}} \varrho_{e x t}+\frac{\sigma}{\varepsilon_{0}} n-D \triangle n=0 \tag{16}
\end{equation*}
$$

This is a modified diffusion equation. In stationary state $\partial_{t} n=0$, and so

$$
\begin{equation*}
\left(\triangle-\frac{\sigma}{D \varepsilon_{0}}\right) n=\frac{\sigma}{D q \varepsilon_{0}} \varrho_{e x t} \tag{17}
\end{equation*}
$$

This is a Helmholtz-equation. Its Green's function, ie. the solution for a point charge is

$$
\begin{equation*}
n_{p}(r)=-\frac{e^{-\kappa r}}{4 \pi r}, \quad \kappa^{2}=\frac{\sigma}{D \varepsilon_{0}} \tag{18}
\end{equation*}
$$

so for an arbitrary external charge distribution we have

$$
\begin{equation*}
n(\boldsymbol{x})=-\frac{\sigma}{D q 4 \pi \varepsilon_{0}} \int d^{3} \boldsymbol{x}^{\prime} \frac{\varrho_{e x t}\left(\boldsymbol{x}^{\prime}\right) e^{-\kappa\left|\boldsymbol{x}-\boldsymbol{x}^{\prime}\right|}}{\left|\boldsymbol{x}-\boldsymbol{x}^{\prime}\right|} \tag{19}
\end{equation*}
$$

We remark that negative density simply means that the ions run out from the system, and the not moving opposite charge background particles remain.

If we want to know the potential of the point particle, we use that in equilibrium, using $\boldsymbol{E}=-\boldsymbol{\nabla} \Phi$ we have

$$
\begin{equation*}
\triangle(\sigma \Phi+q D n)=0 \tag{20}
\end{equation*}
$$

With the boundary condition that in the infinity the potential as well as the particle density is zero we have

$$
\begin{equation*}
\Phi=-\frac{q D}{\sigma} n=\frac{1}{4 \pi \varepsilon_{0}} \int d^{3} \boldsymbol{x}^{\prime} \frac{\varrho_{e x t}\left(\boldsymbol{x}^{\prime}\right) e^{-\kappa\left|\boldsymbol{x}-\boldsymbol{x}^{\prime}\right|}}{\left|\boldsymbol{x}-\boldsymbol{x}^{\prime}\right|} \tag{21}
\end{equation*}
$$

For a point particle with charge $Q$ we have

$$
\begin{equation*}
\Phi=\frac{Q e^{-\kappa r}}{4 \pi \varepsilon_{0} r} \tag{22}
\end{equation*}
$$

This is a screened potential (Yukawa potential) that can be interpreted as the original potential with a distance dependent charge:

$$
\begin{equation*}
Q(r)=Q e^{-\kappa r}, \quad \Phi=\frac{Q(r)}{4 \pi \varepsilon_{0} r} \tag{23}
\end{equation*}
$$

The electric field is the negative gradient of this potential

$$
\begin{equation*}
\boldsymbol{E}=\boldsymbol{e}_{r} \frac{Q(1+\kappa r) e^{-\kappa r}}{4 \pi \varepsilon_{0} r^{2}} \tag{24}
\end{equation*}
$$

If we identify the running coupling with measuring the electric field - after all this is the source of the force - then we would find

$$
\begin{equation*}
Q(r)=Q(1+\kappa r) e^{-\kappa r} \tag{25}
\end{equation*}
$$

### 2.3 Screening by dipoles

In the previous subsection we considered screening caused by moving charges. Here we study the case when it comes from dipoles.

The Maxwell equation is the same as before

$$
\begin{equation*}
\operatorname{div} \boldsymbol{E}=\frac{1}{\varepsilon_{0}}\left(\varrho_{e x t}+\varrho\right) \tag{26}
\end{equation*}
$$

but $\varrho$ is not explicit charge, but it comes from the dipole density as $\varrho=-\operatorname{div} \boldsymbol{P}$. This means that we can rewrite the Maxwell equaiton as

$$
\begin{equation*}
\operatorname{div}\left(\varepsilon_{0} \boldsymbol{E}+\boldsymbol{P}\right)=\varrho_{e x t} \tag{27}
\end{equation*}
$$

this is the equation we know as "Maxwell equation in matter". The $\boldsymbol{P}$ polarization is proportional to $\boldsymbol{E}$ if the system is linear - as it was supposed in the previous subsection. Assuming $\boldsymbol{P}=\varepsilon_{0} \chi \boldsymbol{E}$ we find for the electric field of a point charge

$$
\begin{equation*}
\boldsymbol{E}=\frac{Q}{4 \pi(1+\chi) \varepsilon_{0} r} \tag{28}
\end{equation*}
$$

In electrodynamics this equation is interpreted that the matter changes the dielectric constant $\varepsilon_{0} \rightarrow \varepsilon=$ $(1+\chi) \varepsilon_{0}$. According to the presvious subsections, however, we can interpret this result as a "running" charge

$$
\begin{equation*}
Q(r)=\frac{Q}{1+\chi} \tag{29}
\end{equation*}
$$

This does not depend on the distance, as we see, but in spirit it is in relation with the distance dependent charge. It is also interesting that even at $r \rightarrow 0$ we do not see the original charge, as opposed to the previous cases.

Another interesting phenomenon coming from dipole screening is magnetic screening when the matter has an intrinsic magnetic moment (spin). In the magnetic case an external magnetic field is screened as

$$
\begin{equation*}
\boldsymbol{B}_{\text {screened }}=\boldsymbol{B}(1+\chi) \tag{30}
\end{equation*}
$$

In diamagnets $\chi<0$ and the external magnetic field is screened, as before. But in paramagnets $\chi>0$ and the magnetic field is larger then the external source. Instead of diminishing the external source, therefore, paramagnetic "screening" makes it stronger. This is the "antiscreening" phenomenon, happens also in other cases like in QCD.

### 2.4 Dynamical screening

Let us return to the first case, the screening of the light source. The equation governing the screening is (6); if we write it for $\alpha$ we find

$$
\begin{equation*}
\frac{d \alpha}{d r}=-\sigma n \alpha \tag{31}
\end{equation*}
$$

We have a solution if $n(r)$ is explicitly given. But in realistic cases $n$ is determined also by a dynamical process. For example if the screening caused by insects, the insects themselves is attracted by the light, they are denser in the brighter spots. So let us assume that $n \sim \alpha$; a particularly interesting case comes when

$$
\begin{equation*}
n=C \frac{\alpha}{r} \tag{32}
\end{equation*}
$$

where $C$ is some constant. In this case we have the differential equation

$$
\begin{equation*}
\frac{d \alpha}{d r}=-C \sigma \frac{\alpha^{2}}{r} \tag{33}
\end{equation*}
$$

This is a dynamical, nonlinear screening equation very common in self-screening systems like quantum field theory or statistical physical systems, discussed later in this note. Its solution is

$$
\begin{equation*}
\frac{d}{d \ln r} \frac{1}{\alpha}=C \sigma \quad \Rightarrow \quad \frac{1}{\alpha(r)}-\frac{1}{\alpha\left(r_{0}\right)}=C \sigma \ln \frac{r}{r_{0}} \tag{34}
\end{equation*}
$$

therefore

$$
\begin{equation*}
\alpha(r)=\frac{\alpha\left(r_{0}\right)}{1+\alpha\left(r_{0}\right) C \sigma \ln \frac{r}{r_{0}}} . \tag{35}
\end{equation*}
$$

For large distances it is just logarithmically decreasing funtion - indeed, there the density of the screening insects is smaller than in the constant density case. At small distances, however, we are faced with a new phenomenon, namely

$$
\begin{equation*}
\lim _{r \rightarrow r_{L}} \alpha(r)=\infty, \quad r_{L}=r_{0} e^{-\frac{1}{\alpha\left(r_{0}\right) C \sigma}} \tag{36}
\end{equation*}
$$

In fact we can write the equation for $\alpha$ as

$$
\begin{equation*}
\alpha(r)=\frac{1}{C \sigma \ln \frac{r}{r_{L}}} \tag{37}
\end{equation*}
$$

This divergence is called Landau-pole in the self-interacting cases. Measuring the apparent brightness of the lamp allows us to determine $r_{L}$ when we know the value of the constants $C$ and $\sigma$. But then we know that if we go closer to the source we see brighter and brighter source, and at some point the brightness is infinite. This is certainly an unphysical situation: the only lesson we learn is that our model for screening is incorrect. We should remember this lesson later when we encounter Landau poles in QFT.

### 2.5 Multi-body interactions

To close this section we qualitatively discuss a new phenomenon caused by matter if dynamical screening is present. Let us assume that we examine the light of a lamp through a tube that excludes the light of all other light sources. In vacuum, the light of other sources are simply unobservable; in case of dust the scattered light is visible, but the light of the different sources simply add. In the linear cases the observed light is therefore the sum of the light coming from different sources.

This situation changes when screening is dynamical. Imagine what happens when we switch on a strong reflector placed by side. Even if we do not observe the light of the reflector we know that it is switched on, becasue the flying insects start to drift toward it and the air is cleared, and thus we see the original light source brighter! This means that there is a correlated effect of different sources, they are not independent. The dynamical screening makes the originally linear system nonlinear.

This interesting phenomenon is not rare in nature, in fact it is very common, as we will see later. In electrodynamics the presence of nonlinear screening of matter can cause light scattering, and it can lead to nonlinear optical systems. The same effect works in transistors: there the conducting properties of a semiconductor depend on the number of available charges, similarly than the screening property of the air depends on the available number of insects. If with an approprate field we reduce or even increase the number of available charges in a semiconductor, the resistance of the system changes. In case of a transistor the potential applied on the basis changes the conduction between the emittor and collector, which can be used very efficiently in different applications.

## 3 Master formula for self-interacting systems: path integral

If we want to answer to that question that what is the potential like in an electrolyte, where there are no external charges that would orient the movement of the individual ions, we should know the position of each ion. This is of course impossible, but it is not even necessary: it is enough if we can tell something statistically. The first thought is that we assume that there is an average screened potential, and all ions produce that potential. Then we could write up a self-consistent equation for that potential. This is the mean field approximation, and in a lot of cases it is a good strategy. But in lot of cases a more precise description of the position of the ions is necessary: fluctuations around the mean field can play an important role.

Quantum field theory (QFT) is a genuine self-interacting system, even when we consider only a single particle. The vacuum, namely, is not empty from the point of view of QFT, virtual particle-antiparticle pairs, virtual photons and other gauge bosons show up and disappear. The uncertainty principle that connects the energy and lifetime makes this vivid picture possible: for a time $\Delta t$ the variation of energy is $\sim \hbar / \Delta t$, which means that for short enough time any large energy, and so any particle with arbitrary large mass may appear - and disappear after $\Delta t$ time. The vacuum fluctuations play the same role as the ions in the electorlyte: to tell for example the electric potential of a point charge, the position of all of the particles popping up from vacuum is necessary to know. Again, here we should make statistical statements, just like in the many-body systems.

After realizing the similarity between the many-body systems and quantum field theories it is not a surprise that both can be treated with the same tool: the path integral.

### 3.1 Path integral in statistical physics

In statistical physics in canonical ensemble we evaluate expectation values by taking into account that the probability that we are at $E$ energy level is proportional to $e^{-\beta E}$. So if we consider microstates denoted by $\Phi$, we can compute the average of any $\mathcal{A}(\Phi)$ quantity as

$$
\begin{equation*}
\langle\mathcal{A}\rangle=\frac{1}{Z} \sum_{\Phi} \mathcal{A}(\Phi) e^{-\beta H(\Phi)}, \quad Z=\sum_{\Phi} e^{-\beta H(\Phi)} \tag{38}
\end{equation*}
$$

The summation goes over all possible microstates. This is the master formula of the statistical physics.
There are very lot of possible statistical physical models, but in this note we will restrict ourselves to those models where the microstates are of the form of some field configurations - also to maintain the similarity to QFT models. So we will assume that there is a mesh indexed by $i$, and the microstate is a set $\Phi_{i} \in \boldsymbol{R}$ (or any other vector space). The the summation over all microstates become integrals

$$
\begin{equation*}
\sum_{\Phi}=\int\left(\prod_{i} d \Phi_{i}\right) \equiv \int \mathcal{D} \Phi \tag{39}
\end{equation*}
$$

where the rightmost expression is a notation. Therefore we have

$$
\begin{equation*}
\langle\mathcal{A}\rangle=\frac{1}{Z} \int \mathcal{D} \Phi \mathcal{A}(\Phi) e^{-S(\Phi)}, \quad Z=\int \mathcal{D} \Phi e^{-S(\Phi)}, \quad S(\Phi)=\beta H(\Phi) \tag{40}
\end{equation*}
$$

### 3.2 Path integral in quntum field theories

In quantum field theories we also use field theories with field $\Phi$ and the dynamics is governed by the Lagrangian $L$. To be safe, we assume that the theory is defined also on a mesh, so in fact the configuration is the same as above $\Phi_{i} \in \boldsymbol{R}$. In the course of quantization we replace the values of the fields $\Phi_{i}$ by operators that measure the value of the fields $\hat{\Phi}_{i}$ - these will be called field operator. We will work with bosonic field theories almost for all the time, the fermionic case will be discussed at a later section.

In the operator formalism we compute expectation values by tracing an operator with the density matrix $\varrho$ :

$$
\begin{equation*}
\langle\hat{\mathcal{A}}\rangle=\operatorname{Tr} \hat{\varrho} \hat{\mathcal{A}} \tag{41}
\end{equation*}
$$

In this note we will use vacuum expectation values, which means that $\hat{\varrho}=|0\rangle\langle 0|$. It is possible to write up path integrals for any initial density matrix, thus making possible a formal description of time dependent quantum statistics, but this description goes beyond the framework of this note. The operator for that we want to compute the expectation value will be a time-ordered correlation function of the field operator: $\mathcal{A}=\mathrm{T} \hat{\Phi}_{i_{1}}\left(t_{1}\right) \ldots \hat{\Phi}_{i_{n}}\left(t_{n}\right)$. Time ordering means that we reorganize the original product in a way that the leftmost operator has the largest time argument, the rightmost the smallest, and the time arguments are monotonously growing from right to left.

The time dependence of a single operator comes from the time evolution operator $e^{-i \hat{H} t / \hbar}$, where $\hat{H}$ is the Hamiltonian operator of the system. Later on we will use $\hbar=1$ units, which is just obtained by redefinition of the energy. We will represent the time evolution operator as an integral. First we represent the unit matrix as

$$
\begin{equation*}
\mathbf{1}=\int d \varphi d \pi|\varphi\rangle\langle\varphi \mid \pi\rangle\langle\pi| \tag{42}
\end{equation*}
$$

where $|\varphi\rangle$ is the simultaneous eigenvector of $\hat{\Phi}_{i}\left(t_{0}\right) \equiv \hat{\Phi}_{i}$ and $|\pi\rangle$ is the simultaneous eigenvector of $\hat{\Pi}_{i}\left(t_{0}\right) \equiv \hat{\Pi}_{i}$ for all $i$ :

$$
\begin{equation*}
\hat{\Phi}_{i}|\varphi\rangle=\varphi_{i}|\varphi\rangle, \quad \hat{\Pi}_{i}|\pi\rangle=\pi_{i}|\pi\rangle . \tag{43}
\end{equation*}
$$

This is possible since $\left[\hat{\Phi}_{i}\left(t_{0}\right), \hat{\Phi}_{j}\left(t_{0}\right)\right]=0$ and $\left[\hat{\Pi}_{i}\left(t_{0}\right), \hat{\Pi}_{j}\left(t_{0}\right)\right]=0$ according to the equal time commutation relations. We also know that

$$
\begin{equation*}
\langle\varphi \mid \pi\rangle=e^{i \sum_{i} \phi_{i} \pi_{i}} \equiv e^{i \phi \pi} \tag{44}
\end{equation*}
$$

which is the consequence of $\left[\hat{\Phi}_{i}\left(t_{0}\right), \hat{\Pi}_{j}\left(t_{0}\right)\right]=i \delta_{i j}$. Then we write

$$
\begin{align*}
& e^{-i \hat{H} t}=e^{-i \hat{H} d t} e^{-i \hat{H} d t} \ldots e^{-i \hat{H} d t}= \\
& =\int \prod_{a=0}^{N} d \varphi_{a} \prod_{b=1}^{N} d \pi_{b}\left|\varphi_{0}\right\rangle\left\langle\varphi_{0} \mid \pi_{1}\right\rangle\left\langle\pi_{1}\right| e^{-i \hat{H} d t}\left|\varphi_{1}\right\rangle\left\langle\varphi_{1} \mid \pi_{2}\right\rangle\left\langle\pi_{2}\right| e^{-i \hat{H} d t}\left|\varphi_{2}\right\rangle \ldots \\
& \quad\left\langle\varphi_{N-1} \mid \pi_{N}\right\rangle\left\langle\pi_{N}\right| e^{-i \hat{H} d t}\left|\varphi_{N}\right\rangle\left\langle\varphi_{N}\right| . \tag{45}
\end{align*}
$$

For small enough $d t$ we have up to $\mathcal{O}\left(d t^{2}\right)$

$$
\begin{equation*}
\langle\pi| e^{-i \hat{H} d t}|\varphi\rangle=\langle\pi|(1-i \hat{H} d t)|\varphi\rangle=\langle\pi \mid \varphi\rangle\left(1-i d t \frac{\langle\pi| \hat{H}|\varphi\rangle}{\langle\pi \mid \varphi\rangle}\right)=e^{-i \pi \varphi-i H(\pi, \varphi) d t} \tag{46}
\end{equation*}
$$

Therefore

$$
\begin{align*}
e^{-i \hat{H} t} & =\int \prod_{a=0}^{N} d \varphi_{a} \prod_{b=1}^{N} d \pi_{b}\left|\varphi_{0}\right\rangle e^{i \varphi_{0} \pi_{1}-i \varphi_{1} \pi_{1}-i H\left(\pi_{1}, \varphi_{1}\right) d t+i \varphi_{1} \pi_{2}-i \varphi_{2} \pi_{2}-i H\left(\pi_{2}, \varphi_{2}\right)+\ldots d t}\left\langle\varphi_{N}\right|= \\
& =\int \prod_{a=0}^{N} d \varphi_{a} \prod_{b=1}^{N} d \pi_{b}\left|\varphi_{0}\right\rangle e^{i \sum_{a=1}^{N}\left(\partial_{t} \varphi_{a} \pi_{a}-H\left(\pi_{a}, \varphi_{a}\right)\right)}\left\langle\varphi_{N}\right| \tag{47}
\end{align*}
$$

where we denoted $\partial_{t} \varphi_{a} d t=\varphi_{a-1}-\varphi_{a}$. It is in fact just a notation, since it is not guaranteed that $\varphi(t)=\varphi(a d t)$ as $d t \rightarrow 0$ goes to a differentiable funtcion - usually it does not. But with this notation we can recover

$$
\begin{equation*}
\partial_{t} \varphi \pi-H(\pi, \varphi)=L\left(\pi, \partial_{t} \varphi, \varphi\right) \tag{48}
\end{equation*}
$$

and the $d t \rightarrow 0$ limit then goes to a Riemann-integral. Since $\int d t L=S$ we find

$$
\begin{equation*}
e^{-i \hat{H} t}=\int \prod_{a=0}^{N} d \varphi_{a} \prod_{b=1}^{N} d \pi_{b}\left|\varphi_{0}\right\rangle e^{i S\left[\pi, \partial_{t} \varphi, \varphi\right]}\left\langle\varphi_{N}\right| \tag{49}
\end{equation*}
$$

Usually the conjugate momentum dependent part of $H$ is $\sim \pi^{2}$, then the $\pi$ integrals can be performed:

$$
\begin{equation*}
\int d \pi e^{i\left(\partial_{t} \varphi \pi-\frac{1}{2} \pi^{2}\right)}=\int d \pi e^{-\frac{i}{2}\left(\pi-\partial_{t} \varphi\right)^{2}+\frac{i}{2}\left(\partial_{t} \varphi\right)^{2}}=C e^{\frac{i}{2}\left(\partial_{t} \varphi\right)^{2}} \tag{50}
\end{equation*}
$$

In this case we recover the classical action and we can write

$$
\begin{equation*}
e^{-i \hat{H} t}=\int \mathcal{D} \varphi\left|\varphi_{0}\right\rangle e^{i S}\left\langle\varphi_{N}\right| \tag{51}
\end{equation*}
$$

With the help of this representation we can calculate the correlation functions of the quantum fields, too. For the time dependent fields we have

$$
\begin{equation*}
\hat{\Phi}_{i}(t)=e^{i \hat{H}\left(t-t_{0}\right)} \hat{\Phi}_{i} e^{-i \hat{H}\left(t-t_{0}\right)}, \quad \hat{\Phi}_{i}=\hat{\Phi}_{i}\left(t_{0}\right) \tag{52}
\end{equation*}
$$

Let us now consider a time ordered series $t_{n}<t_{n-1}<\cdots<t_{1}$, then

$$
\begin{align*}
& \mathrm{T} \hat{\Phi}_{i_{1}}\left(t_{1}\right) \ldots \hat{\Phi}_{i_{n}}\left(t_{n}\right)=\hat{\Phi}_{i_{1}}\left(t_{1}\right) \ldots \hat{\Phi}_{i_{n}}\left(t_{n}\right)= \\
& =e^{i \hat{H}\left(t_{1}-t_{0}\right)} \hat{\Phi}_{i_{1}} e^{-i \hat{H}\left(t_{1}-t_{2}\right)} \hat{\Phi}_{i_{2}} e^{-i \hat{H}\left(t_{2}-t_{3}\right)} \ldots e^{-i \hat{H}\left(t_{n-1}-t_{n}\right)} \hat{\Phi}_{i_{n}} e^{-i \hat{H}\left(t_{n}-t_{0}\right)} \tag{53}
\end{align*}
$$

Now we write into this formula the path integral representation of the time evolution operator, and use that $\langle\varphi| \hat{\Phi}\left|\varphi^{\prime}\right\rangle=\varphi \delta\left(\varphi-\varphi^{\prime}\right)$. We also extend the integrals with time intervals from $t_{1} \rightarrow t_{f}$ and $t_{i} \rightarrow t_{0}$. Finally the different representations can be merged together to find

$$
\begin{equation*}
e^{i \hat{H}\left(t_{f}-t_{0}\right)} \int \mathcal{D} \varphi\left|\varphi_{0}\right\rangle e^{i \int_{t_{i}}^{t_{f}} d t \mathcal{L}} \varphi_{i_{1}}\left(t_{1}\right) \ldots \varphi_{i_{n}}\left(t_{n}\right)\left\langle\varphi_{N}\right| e^{i \hat{H}\left(t_{0}-t_{i}\right)} \tag{54}
\end{equation*}
$$

If we take the vacuum expectation value we can use the fact that

$$
\begin{equation*}
e^{i \hat{H} t}|0\rangle=Z|0\rangle \tag{55}
\end{equation*}
$$

since the vacuum is intact under the time evolution. We find

$$
\begin{equation*}
Z=\langle 0| e^{i \hat{H} t}|0\rangle \tag{56}
\end{equation*}
$$

and so finally

$$
\begin{equation*}
\langle 0| \mathrm{T} \hat{\Phi}_{i_{1}}\left(t_{1}\right) \ldots \hat{\Phi}_{i_{n}}\left(t_{n}\right)|0\rangle=\frac{1}{Z} \int \mathcal{D} \varphi e^{i \int_{t_{i}}^{t_{f}} d t \mathcal{L}} \varphi_{i_{1}}\left(t_{1}\right) \ldots \varphi_{i_{n}}\left(t_{n}\right) \tag{57}
\end{equation*}
$$

We can send $t_{i} \rightarrow-\infty$ and $t_{f} \rightarrow \infty$, then

$$
\begin{equation*}
\langle\mathcal{A}\rangle=\frac{1}{Z} \int \mathcal{D} \varphi \mathcal{A} e^{i S}, \quad Z=\int \mathcal{D} \varphi e^{i S} \tag{58}
\end{equation*}
$$

where the expectation value here denotes vacuum expectation value.
We can see the formal analogy between the statistical physical description and the vacuum expectation value in QFT. The only difference that in one case there is $i S$ that appears in the exponent, in the other case it is $-S$. Formally we can perform a change of variables $t=-i \tau$, then

$$
\begin{equation*}
i \int_{-\infty}^{\infty} d t \mathcal{L}\left(\partial_{t} \varphi, \partial_{i} \varphi, \varphi\right)=\int_{-i \infty}^{i \infty} d \tau \mathcal{L}\left(-i \partial_{\tau} \varphi, \partial_{i} \varphi, \varphi\right)=-\int_{-\infty}^{\infty} d \tau \mathcal{L}_{E}\left(\partial_{\tau} \varphi, \partial_{i} \varphi, \varphi\right) \tag{59}
\end{equation*}
$$

where in the last step we performed a rotation of the integration contour (Wick rotation), and introduced

$$
\begin{equation*}
\mathcal{L}_{E}\left(\partial_{\tau} \varphi, \partial_{i} \varphi, \varphi\right)=-\mathcal{L}\left(-i \partial_{\tau} \varphi, \partial_{i} \varphi, \varphi\right) \tag{60}
\end{equation*}
$$

The $E$ index refers to the "Euclidean" field theory. Then we can compute expectation values with field arguments $x=(-i \tau, \mathbf{x})$ :

$$
\begin{equation*}
\langle\mathcal{A}\rangle_{E}=\frac{1}{Z} \int \mathcal{D} \varphi \mathcal{A} e^{-S_{E}}, \quad Z=\int \mathcal{D} \varphi e^{-S_{E}} \tag{61}
\end{equation*}
$$

This formula is in complete analogy with the result coming from statistical physics 40).
Let us emphasize that although the result formally is written up as a continuum result, we in fact always have a discretization procedure in the background. In fact the continuum limit is very tricky, and naively it does not exists (apart from the most simple cases), only a careful limit, called renormalization that allows to perform the continuum limit.

### 3.3 Generator functional

Now we define some techniques to facilitate the treatment of the expectation values. The first one is the generator functional defined as

$$
\begin{equation*}
Z[J]=\int \mathcal{D} \varphi e^{-S[\varphi]+\int d x J(x) \varphi(x)} \tag{62}
\end{equation*}
$$

where we denote the integration measure by $\int d x$ in arbitrary dimensions. As we emphasized several times, the continuum notation is just a simplification, so in fact we could have written $\sum_{i} J_{i} \varphi_{i}$ instead of the integral. In the followings, however, we will fully omit the notation of summation-integration, and we just write

$$
\begin{equation*}
Z[J]=\int \mathcal{D} \varphi e^{-S[\varphi]+J \varphi}, \tag{63}
\end{equation*}
$$

where multiplication of fields implicitly means integral or sum.
It is worth to mention that formally the above formula is a Laplace-transform, or $Z[i J]$ is the functional Fourier transform of $e^{-S}$. Since Fourier transform can be inverted, this means that the classical action can be obtained from $Z[i J]$ with the same path integral technique.

With this definition the correlation functions can be computed as

$$
\begin{equation*}
\left\langle\Phi_{1} \ldots \Phi_{n}\right\rangle=\left.\frac{1}{Z[J]} \frac{\partial Z[J]}{\partial J_{1} \ldots \partial J_{n}}\right|_{J=0} \tag{64}
\end{equation*}
$$

We will denote the $n$th derivative by

$$
\begin{equation*}
\frac{\partial Z[J]}{\partial J_{1} \ldots \partial J_{n}}=Z_{1 \ldots n}^{(n)}[J] \tag{65}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
\left\langle\Phi_{1} \ldots \Phi_{n}\right\rangle=\frac{Z_{1 \ldots n}^{(n)}[J=0]}{Z[J=0]} \tag{66}
\end{equation*}
$$

$Z[J]$ can be interpreted in statistical physics as the partition function of a theory with the Hamiltonian

$$
\begin{equation*}
H=H_{0}-T J \varphi \tag{67}
\end{equation*}
$$

For example if we have a spin system in external magnetic field then $H=H_{0}-\boldsymbol{B} \boldsymbol{m}$. So $Z[J]$ is the partition function in presence of some external effect.

## $3.4 \quad \Phi^{4}$ model

The usual trial field theory, where one can the most easily demonstrate how the different field theoretical methods work, is the $\Phi^{4}$ model. In the continuum notation the basic field is a $\Phi: \boldsymbol{R}^{4} \rightarrow \boldsymbol{R}$ scalar field $\Phi:(t, \mathbf{x}) \mapsto \Phi(t, \mathbf{x})$. Although we often use this comfortable notation, but the formulae are sensible only in a discretized version. This can be $\Phi_{i}(t)$, where $i$ indexes a space-mesh (the lattice), but, as we will see later, a discretization can be completely different. For example we can apply a momentum cutoff, where $\Phi(p)=0$ for $|\boldsymbol{p}|^{2}>\Lambda^{2}$ either for the 3D momentum, or the 4D Wick-rotated momentum.

The Lagrangian reads

$$
\begin{equation*}
\mathcal{L}=\frac{1}{2}\left(\partial_{\mu} \Phi\right)\left(\partial^{\mu} \Phi\right)-U(\Phi) \tag{68}
\end{equation*}
$$

where the potential $U$ is

$$
\begin{equation*}
U=\frac{m^{2}}{2} \Phi^{2}+\frac{\lambda}{24} \Phi^{4} \tag{69}
\end{equation*}
$$

This system has a $Z_{2}$ symmetry $\Phi \rightarrow-\Phi$. For a sensible theory, the potential must be bounded from below which means $\lambda>0$.

The canoniclly conjugated momentum reads

$$
\begin{equation*}
\Pi=\frac{\partial \mathcal{L}}{\partial \partial_{t} \Phi}=\partial_{t} \Phi \tag{70}
\end{equation*}
$$

and so the Hamiltonian density

$$
\begin{equation*}
\mathcal{H}=\Pi \partial_{t} \Phi-\mathcal{L}=\frac{1}{2} \Pi^{2}+\frac{1}{2}\left(\partial_{i} \Phi\right)^{2}+\frac{m^{2}}{2} \Phi^{2}+\frac{\lambda}{24} \Phi^{4} \tag{71}
\end{equation*}
$$

We can treat this model classically, which means that we start from an initial condition (or an ensemble of initial conditions) and solve the classical equation of motions (EoM):

$$
\begin{equation*}
\left(\partial^{2}+m^{2}+\frac{\lambda}{6} \Phi^{2}\right) \Phi=0 . \tag{72}
\end{equation*}
$$

Minima of the energy characterize the ground state of the system either in statistical physics (energy or free energy) or in field theory, including QFT. Since the gradient term always increases the energy, the minimal energy is reached for constant configurations. For constant $\Phi(t, \mathbf{x})=\Phi_{0}$ configurations the EoM reads:

$$
\begin{equation*}
\left(m^{2}+\frac{\lambda}{6} \Phi_{0}^{2}\right) \Phi_{0}=0 \tag{73}
\end{equation*}
$$

The $\Phi_{0}=0$ is always a solution: this corresponds to the symmetric solution, symmetric vacuum. If $m^{2}<0$ then

$$
\begin{equation*}
\Phi_{0}= \pm \frac{-6 m^{2}}{\lambda} \tag{74}
\end{equation*}
$$

is also a solution. This is the spontaneously symmetry broken (SSB) case, since the $Z_{2}$ symmetry is not manifested in the $\Phi_{0}$ solution, but the $Z_{2}$ brings one solution to the other. The second derivative of the potential

$$
U^{\prime \prime}=m^{2}+\frac{\lambda}{2} \Phi^{2}= \begin{cases}m^{2}, & \Phi=0  \tag{75}\\ -2 m^{2}, & \Phi_{0}= \pm \frac{-6 m^{2}}{\lambda}\end{cases}
$$

which means that the minimum is at $\Phi=0$ for $m^{2}>0$, and at $\Phi_{0}$ for $m^{2}<0$.
If we start from a non-constant configuration, after a certain time the system reaches its thermal state, then we can define an equilibrium statistical physical system. If we are interested in time average of spatial correlation functions

$$
\begin{equation*}
\left\langle\Phi\left(\mathbf{x}_{1}\right) \ldots \Phi\left(\mathbf{x}_{n}\right)\right\rangle=\lim _{T \rightarrow \infty} \frac{1}{T} \int_{t_{0}}^{t_{0}+T} d t^{\prime} \Phi\left(t^{\prime}, \mathbf{x}_{1}\right) \ldots \Phi\left(t^{\prime}, \mathbf{x}_{n}\right) \tag{76}
\end{equation*}
$$

then it can be computed using the ensemble of configurations as follows. We define

$$
\begin{equation*}
\mathcal{P}[\varphi, \pi]=\lim _{T \rightarrow \infty} \frac{1}{T} \int_{t_{0}}^{t_{0}+T} d t^{\prime} \prod_{\mathbf{x}} \delta\left(\varphi(\mathbf{x})-\Phi\left(t^{\prime}, \mathbf{x}\right)\right) \delta\left(\pi(\mathbf{x})-\Pi\left(t^{\prime}, \mathbf{x}\right)\right) \tag{77}
\end{equation*}
$$

distribution function (histogram). With the help of this function we can change from

$$
\begin{align*}
\left\langle\Phi\left(\mathbf{x}_{1}\right) \ldots \Phi\left(\mathbf{x}_{n}\right)\right\rangle & =\lim _{T \rightarrow \infty} \frac{1}{T} \int_{t_{0}}^{t_{0}+T} \Phi\left(t^{\prime}, \mathbf{x}_{1}\right) \ldots \Phi\left(t^{\prime}, \mathbf{x}_{n}\right) \int \mathcal{D} \varphi \mathcal{D} \pi \delta\left(\varphi(\mathbf{x})-\Phi\left(t^{\prime}, x\right)\right) \delta\left(\pi(\mathbf{x})-\Pi\left(t^{\prime}, x\right)\right) \\
& =\int \mathcal{D} \varphi \mathcal{D} \pi \varphi\left(\mathbf{x}_{1}\right) \ldots \varphi\left(\mathbf{x}_{n}\right) \lim _{T \rightarrow \infty} \frac{1}{T} \int_{t_{0}}^{t_{0}+T} \delta\left(\varphi(\mathbf{x})-\Phi\left(t^{\prime}, x\right)\right) \delta\left(\pi(\mathbf{x})-\Pi\left(t^{\prime}, x\right)\right)= \\
& =\int \mathcal{D} \varphi \mathcal{D} \pi \mathcal{P}[\varphi, \pi] \varphi\left(\mathbf{x}_{1}\right) \ldots \varphi\left(\mathbf{x}_{n}\right) \tag{78}
\end{align*}
$$

Although it is a dynamical question, in most cases the distribution function is Boltzmannian, ie. $\mathcal{P}[\varphi, \pi] \sim$ $e^{-\beta H[\varphi, \pi]}$ with an appropriate $\beta$ coming from energy conservation, and with an appropriate normalization factor. Then we have

$$
\begin{equation*}
\left\langle\Phi\left(\mathbf{x}_{1}\right) \ldots \Phi\left(\mathbf{x}_{n}\right)\right\rangle=\frac{1}{Z_{\pi}} \int \mathcal{D} \varphi \mathcal{D} \pi e^{-\beta H[\varphi, \pi]} \varphi\left(\mathbf{x}_{1}\right) \ldots \varphi\left(\mathbf{x}_{n}\right) \tag{79}
\end{equation*}
$$

If we are not interested in $\Pi$ expectation values, then we can integrate over the $\pi$ variables and obtain

$$
\begin{equation*}
S[\varphi]=\beta \int d^{3} \mathbf{x}\left[\frac{1}{2}\left(\partial_{i} \Phi\right)^{2}+\frac{m^{2}}{2} \Phi^{2}+\frac{\lambda}{24} \Phi^{4}\right] \tag{80}
\end{equation*}
$$

with that function

$$
\begin{equation*}
\left\langle\Phi\left(\mathbf{x}_{1}\right) \ldots \Phi\left(\mathbf{x}_{n}\right)\right\rangle=\frac{1}{Z} \int \mathcal{D} \varphi e^{-S[\varphi]} \varphi\left(\mathbf{x}_{1}\right) \ldots \varphi\left(\mathbf{x}_{n}\right) \tag{81}
\end{equation*}
$$

If $\Phi^{4}$ model is treated as a quantum field theory, then the vacuum expectation value of a correlation function reads

$$
\begin{equation*}
\langle 0| \mathrm{T} \hat{\Phi}\left(x_{1}\right) \ldots \hat{\Phi}\left(x_{n}\right)|0\rangle=\frac{1}{Z} \int \mathcal{D} \varphi e^{-i S[\varphi]} \varphi\left(x_{1}\right) \ldots \varphi\left(x_{n}\right), \tag{82}
\end{equation*}
$$

where $x=(t, \mathbf{x})$ spacetime coordinates. After Wick rotation we have

$$
\begin{equation*}
S_{E}=\int d^{3} \mathbf{x} d \tau\left[\frac{1}{2}\left(\partial_{\tau} \Phi\right)^{2}+\frac{1}{2}\left(\partial_{i} \Phi\right)^{2}+\frac{m^{2}}{2} \Phi^{2}+\frac{\lambda}{24} \Phi^{4}\right] \tag{83}
\end{equation*}
$$

We can still compute the correlation functions with $x=(-i \tau, \mathbf{x})$ arguments

$$
\begin{equation*}
\left\langle\hat{\Phi}\left(x_{1}\right) \ldots \hat{\Phi}\left(x_{n}\right)\right\rangle_{E}=\frac{1}{Z} \int \mathcal{D} \varphi e^{-S_{E}[\varphi]} \varphi\left(x_{1}\right) \ldots \varphi\left(x_{n}\right) \tag{84}
\end{equation*}
$$

From here the original correlation functions can be obtained with analytic continuation.
We see that the Wick rotated $3+1$ dimensional quantum $\Phi^{4}$ theory is equivalent a classical model in $4+1$ dimensions - this correspondance is usually true. In the followings we therefore study Euclidean $\Phi^{4}$ models in arbitrary dimensions.

## 4 Perturbation theory

### 4.1 Free theories

We can compute explicitly $Z[J]$ in case of quadratic models. In this case

$$
\begin{equation*}
S=\frac{1}{2} \Phi_{i} \mathcal{K}_{i j} \Phi_{j} \tag{85}
\end{equation*}
$$

with some real symmetric kernel $\mathcal{K}_{i j}=\mathcal{K}_{j i}$. For the scalar model, after an appropriate partial integration

$$
\begin{equation*}
S=\int d^{4} x \frac{1}{2} \Phi(x)\left(-\partial^{2}+m^{2}\right) \Phi(x)=\int \frac{d^{4} p}{(2 \pi)^{4}} \frac{1}{2} \Phi^{*}(p)\left(p^{2}+m^{2}\right) \Phi(p) \tag{86}
\end{equation*}
$$

where $\Phi^{*}(p)=\Phi(-p)$ because $\Phi(x)$ is real.
We remark that if $\Phi$ is complex valued, then the real action reads

$$
\begin{equation*}
S=\frac{1}{2} \Phi_{i}^{\dagger} \mathcal{K}_{i j} \Phi_{j} \tag{87}
\end{equation*}
$$

where the kernel is hermitian $\mathcal{K}_{i j}=\mathcal{K}_{j i}^{\dagger}$. By introducing $\Phi=\Phi^{(r)}+i \Phi^{(i)}$ we can write

$$
\begin{equation*}
S=\frac{1}{2}\left(\Phi_{i}^{(r)} \mathcal{K}_{i j} \Phi_{j}^{(r)}+\Phi_{i}^{(i)} \mathcal{K}_{i j} \Phi_{j}^{(i)}\right) \tag{88}
\end{equation*}
$$

which is the same form we started with.
For $J=0$ we find for the partition function

$$
\begin{equation*}
Z_{0}=\int \mathcal{D} \varphi e^{-\frac{1}{2} \varphi_{i} \mathcal{K}_{i j} \varphi_{j}}=(2 \pi)^{N / 2}(\operatorname{det} \mathcal{K})^{-1 / 2} \tag{89}
\end{equation*}
$$

where $N$ is the size of the index set $\{i\}$. Actually, a constant multiplicative factor in $Z$ does not matter, so we can normalize the path integral integration measure to yield

$$
\begin{equation*}
Z_{0}=\int \mathcal{D} \varphi e^{-\frac{1}{2} \varphi_{i} \mathcal{K}_{i j} \varphi_{j}}=(\operatorname{det} \mathcal{K})^{-1 / 2} \tag{90}
\end{equation*}
$$

For the $J$ dependent case we have

$$
\begin{equation*}
Z_{0}[J]=\int \mathcal{D} \varphi e^{-\frac{1}{2} \varphi_{i} \mathcal{K}_{i j} \varphi_{j}+J_{i} \varphi_{i}}=\int \mathcal{D} \varphi e^{-\frac{1}{2}\left(\varphi_{i}-G_{i k} J_{k}\right) \mathcal{K}_{i j}\left(\varphi_{j}-G_{j k} J_{k}\right)+\frac{1}{2} J_{i} G_{i j} J_{j}}=Z_{0} e^{\frac{1}{2} J_{i} G_{i j} J_{j}} \tag{91}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{K}_{i j} G_{j k}=\delta_{j k} \tag{92}
\end{equation*}
$$

i.e. $G_{i j}$ is the inverse of $\mathcal{K}_{i j}$, it is a symmetric matrix. too. This means that $G$ is the propagator of the free equations of motion; indeed the equations of motion in the quadratic model

$$
\begin{equation*}
\mathcal{K}_{i j} \Phi_{j}=J_{i} \quad \Rightarrow \quad \Phi_{i}=G_{i j} J_{j} \tag{93}
\end{equation*}
$$

Computing the 2-point correlation function we obtain

$$
\begin{equation*}
\left\langle\Phi_{i} \Phi_{j}\right\rangle=\left.\frac{1}{Z[J]} \frac{\partial^{2} Z[J]}{\partial J_{i} \partial J_{j}}\right|_{J=0}=G_{i j} \tag{94}
\end{equation*}
$$

so the propagator is also the 2-point correlation function.
The $n$-point function in the quadratic model according to 64

$$
\begin{equation*}
\left\langle\Phi_{1} \ldots \Phi_{n}\right\rangle=\left.e^{-\frac{1}{2} J_{i} G_{i j} J_{j}} \frac{\partial^{n}}{\partial J_{1} \ldots \partial J_{n}} e^{\frac{1}{2} J_{i} G_{i j} J_{j}}\right|_{J=0}=\sum_{i \text { pairings }} G_{i_{1} i_{2}} G_{i_{3} i_{4}} \ldots G_{i_{n-1} i_{n}} \tag{95}
\end{equation*}
$$

since one $J$ derivative brings down a $G J$ term which has to be vanished by some other derivative otherwise the $J=0$ condition sends it to zero. This formula is known as the Wick theorem.

In the scalar model the kernel reads

$$
\begin{equation*}
\mathcal{K}=p^{2}+m^{2} \tag{96}
\end{equation*}
$$

therefore, using $\ln \operatorname{det}=\operatorname{Tr} \ln$

$$
\begin{equation*}
-\ln Z_{0}=\beta F=\frac{1}{2} \int \frac{d^{d} p}{(2 \pi)^{d}} \ln \left(p^{2}+m^{2}\right) \tag{97}
\end{equation*}
$$

where $F$ the (formal) free energy. We see that the integral is not convergent in the continuum limit - we will return to this question.

The propagator of the quadratic model is

$$
\begin{equation*}
G(p)=\frac{1}{p^{2}+m^{2}} \tag{98}
\end{equation*}
$$

We remark that in the Minkowskian space the propagator is not unique, because of the poles at $p^{2}=m^{2}$. According to the way we go around the poles we can have different propagators, as

$$
\begin{equation*}
G_{F}=\frac{1}{p^{2}-m^{2}+i \varepsilon}, \quad G_{r e t}=\frac{1}{\left(p_{0}+i \varepsilon\right)^{2}-\boldsymbol{p}^{2}+m^{2}} \tag{99}
\end{equation*}
$$

### 4.2 Interactions

We have seen that in the quadratic model we can easily calculate the functional generators. Unfortunately for a generic action we can not perform the functional integral. We will call non-quadratic theories "interacting", and the non-quadratic terms are the "interactions".

The first thought to treat interacting theories is to assume that the effect of interactions is small, and we can make an expansion around the quadratic models to have the results of interacting ones. The idea is very simple: we write

$$
\begin{equation*}
S[\varphi]=S_{0}[\varphi]+S_{i n t}[\varphi] \tag{100}
\end{equation*}
$$

where $S_{0}$ denotes the quadratic part. Then we write for the path integral

$$
\begin{equation*}
Z[J]=\int \mathcal{D} \varphi e^{-S_{0}[\varphi]-S_{i n t}[\varphi]+J \varphi}=\sum_{m=0}^{\infty} \frac{1}{m!} \int \mathcal{D} \varphi\left(-S_{i n t}[\varphi]\right)^{m} e^{-S_{0}[\varphi]+J \varphi} \tag{101}
\end{equation*}
$$

Since $S_{\text {int }}$ is a polynomial in the field, all terms in the above expansion requires evaluation of free correlation functions, that we already performed in (95). If the interaction has some coefficient - the "coupling constant", like $\lambda / 24$ in the $\Phi^{4}$ model - then the above expansion is a power series in the coupling constant. So we can assume that for small enough coupling the series is convergent, and we can approach the noninteracting result arbitrarily close.

The $n$th derivative of the above expression is the $n$-point functions. They are represented by Feynman diagrams

- we represent a propagator with momentum $p$ by a line; in the momentum representation we assign a $\int \frac{d^{4} p_{a}}{(2 \pi)^{4}} G_{0}\left(p_{a}\right)$ expression to each line
- from external fields we can start a line: these are the incoming momenta; we associate $G_{0}\left(p_{i n}\right)$ term to external lines
- terms coming from $S_{i n t} \sim \Phi^{z}$ connect $z$ lines; these are visualized as vertices in the graph with $z$ lines flowing to it; their value is $(-\lambda)(2 \pi)^{4} \delta\left(\sum p_{a}\right)$
- we write down all possible diagrams with $m$ vertices and $n$ external lines, and evaluate their values according to the rules above
- add up the different Feynman-diagram contributions.

For example we compute the 2-point function in $\Phi^{4}$ theory to order $\lambda^{2}$. We have

$$
\begin{equation*}
\langle\Phi(x) \Phi(y)\rangle=\frac{\left\langle\Phi(x) \Phi(y) e^{-S_{i n t}}\right\rangle_{0}}{\left\langle e^{-S_{i n t}}\right\rangle_{0}} \tag{102}
\end{equation*}
$$

where 0 subscript means quadratic expectation values. The numerator reads:

$$
\begin{align*}
& \left\langle\Phi(x) \Phi(y) e^{-S_{i n t}}\right\rangle= \\
& =\langle\Phi(x) \Phi(y)\rangle-\frac{\lambda}{24} \int d^{4} z\left\langle\Phi(x) \Phi(y) \Phi^{4}(z)\right\rangle+\frac{1}{2}\left(\frac{\lambda}{24}\right)^{2} \int d^{4} z d^{4} w\left\langle\Phi(x) \Phi(y) \Phi^{4}(z) \Phi^{4}(w)\right\rangle+\mathcal{O}\left(\lambda^{3}\right) \tag{103}
\end{align*}
$$

The first term is the free propagator $G_{0}(x-y)$. It depends on $x-y$, like all other terms, because of 4 -space translation symmetry. When we change to Fourier space, we should Fourier transform with respect to $x-y$, the result is $1 /\left(p^{2}+m^{2}\right)$.

The expectation value in the second term reads

$$
\begin{equation*}
\left\langle\Phi(x) \Phi(y) \Phi^{4}(z)\right\rangle=3 G_{0}(x-y) G_{0}^{2}(0)+12 G_{0}(x-z) G_{0}(y-z) G_{0}(0) \tag{104}
\end{equation*}
$$

where the prefactors come from counting the diagrams that give the same contribution. The first contribution comes after connecting $\Phi(x)$ and $\Phi(y)$, and we prepare pairs by connecting the remaining $\Phi^{4}(z)$. Here the first $\Phi(z)$ can be connected to any of the other 3 , the remaining two should be connected to themselves. Thus the factor of 3 . In a similar way in the second contribution $\Phi(x)$ is connected to any of the $\Phi(z)$ in the $\Phi^{4}$ interaction: this can be done in 4 different ways. Then $\Phi(y)$ can be connected to any of the remaining $3 \Phi(z)$ fields, and the remaining two fields are connected to themselves. This together yields a factor of 12. Let us denote

$$
\begin{equation*}
\mathcal{T}=G_{0}(z=0)=\int \frac{d^{4} p}{(2 \pi)^{4}} G_{0}(p)=\int \frac{d^{4} p}{(2 \pi)^{4}} \frac{1}{p^{2}+m^{2}} \tag{105}
\end{equation*}
$$

this is the "tadpole" diagram. The $\mathcal{O}(\lambda)$ terms correspond to the Feynman diagrams of Fig. 1 .


Figure 1: The $\mathcal{O}(\lambda)$ contributions to the 2-point function in $\Phi^{4}$ theory.
The expectation value in the third term

$$
\begin{align*}
& \left\langle\Phi(x) \Phi(y) \Phi^{4}(z) \Phi^{4}(w)\right\rangle=9 G_{0}(x-y) G_{0}^{4}(0)+12 G_{0}(x-y) G_{0}^{2}(w-z) G_{0}^{2}(0)+24 G_{0}(x-y) G_{0}^{4}(w-z)+ \\
& +96 G_{0}(x-z) G_{0}(y-z) G_{0}^{3}(0)+2(12)^{2} G_{0}(x-z) G_{0}(y-z) G_{0}^{2}(z-w) G_{0}(0)+ \\
& +2(12)^{2} G_{0}(x-z) G_{0}(z-w) G_{0}(w-y) G_{0}^{2}(0)+8 \cdot 24 G_{0}(x-z) G_{0}(z-w) G_{0}^{3}(w-y) \tag{106}
\end{align*}
$$

The $\mathcal{O}\left(\lambda^{2}\right)$ terms correspond to the Feynman diagrams depicted in Fig. 2. There are two more diagrams


Figure 2: The $\mathcal{O}\left(\lambda^{2}\right)$ contributions to the 2-point function in $\Phi^{4}$ theory.
that are worth to define. One is the "bubble" diagram

$$
\begin{equation*}
\mathcal{B}_{2}(z)=G_{0}^{2}(z) \tag{107}
\end{equation*}
$$

In Fourier space a product corresponds to convolution:

$$
\begin{equation*}
\mathcal{B}_{2}(p)=\int \frac{d^{4} q}{(2 \pi)^{4}} \frac{1}{\left(q^{2}+m^{2}\right)\left((p-q)^{2}+m^{2}\right)} \tag{108}
\end{equation*}
$$

The other is the "setting sun" diagram

$$
\begin{equation*}
\mathcal{B}_{3}(z)=G_{0}^{3}(z) \tag{109}
\end{equation*}
$$

in Fourier space it correponds to a twofold integral

$$
\begin{equation*}
\mathcal{B}_{3}(p)=\int \frac{d^{4} q}{(2 \pi)^{4}} \frac{d^{4} k}{(2 \pi)^{4}} \frac{1}{\left(q^{2}+m^{2}\right)\left(k^{2}+m^{2}\right)\left((p-q-k)^{2}+m^{2}\right)} \tag{110}
\end{equation*}
$$

This is already a two-loop integral, since there are two momentum integration in this expression. In general we can define

$$
\begin{equation*}
\mathcal{B}_{n}(z)=G_{0}^{n}(z), \quad \mathcal{B}_{n}(p)=\int \prod_{a=1}^{n}\left(\frac{d^{d} q_{a}}{(2 \pi)^{d}} \frac{1}{\left(q_{a}^{2}+m^{2}\right)}\right)(2 \pi)^{d} \delta\left(p-\sum_{a} q_{a}\right) \tag{111}
\end{equation*}
$$

which is an $n-1$-loop integral.
The previous result has to be normalized by $\left\langle e^{-S_{\text {int }}}\right\rangle$, c.f. 102 . The diagrams belonging to this expectation value have no external legs:

$$
\begin{equation*}
\left\langle e^{-S_{i n t}}\right\rangle=1-\frac{\lambda}{24} \int d^{4} z\left\langle\Phi^{4}(z)\right\rangle+\frac{1}{2}\left(\frac{\lambda}{24}\right)^{2} \int d^{4} z d^{4} w\left\langle\Phi^{4}(z) \Phi^{4}(w)\right\rangle+\mathcal{O}\left(\lambda^{2}\right) \tag{112}
\end{equation*}
$$

These are called the vacuum diagrams. We can realize that these diagrams appear as multiplicative factors besides each digrams that are connected to the external lines. After division with the vacuum diagrams only 4 contribution remains

$$
\begin{equation*}
\langle\Phi \Phi\rangle(p)=G_{0}(p)-\frac{\lambda}{2} G_{0}^{2}(p) \mathcal{T}+\frac{\lambda^{2}}{4} G_{0}^{2}(p) \mathcal{B}_{2}(0) \mathcal{T}+\frac{\lambda^{2}}{4} G_{0}^{3}(p) \mathcal{T}^{2}+\frac{\lambda^{2}}{6} G_{0}^{2}(p) \mathcal{B}_{3}(p)+\mathcal{O}\left(\lambda^{3}\right) \tag{113}
\end{equation*}
$$

We can realize that to the same prescision it can be written as

$$
\begin{equation*}
\langle\Phi \Phi\rangle(p)=\frac{1}{p^{2}+m^{2}+\Sigma(p)}=G_{0}(p)-G_{0}^{2}(p) \Sigma(p)+G_{0}^{3}(p) \Sigma^{2}(p)+\ldots \tag{114}
\end{equation*}
$$

where

$$
\begin{equation*}
\Sigma(p)=\frac{\lambda}{2} \mathcal{T}-\frac{\lambda^{2}}{4} \mathcal{B}_{2}(0) \mathcal{T}-\frac{\lambda^{2}}{6} \mathcal{B}_{3}(p)+\mathcal{O}\left(\lambda^{3}\right) \tag{115}
\end{equation*}
$$

$\Sigma$ is called self-energy.
In a more complicated expectation value there are a lot of terms appearing. But some of these diagrams are not independent. To make calculations simpler we introduce other functional methods.

## 5 Functional methods

To facilitate the computation we can use auxiliary quantities. In this section we overview the standard ones.

### 5.1 Free energy

The partition function is related to the free energy as $Z=e^{-\beta F}$, so it is worth to define a free energy-like quantity

$$
\begin{equation*}
Z[J]=e^{W[J]} \tag{116}
\end{equation*}
$$

The expectation value of the field operator

$$
\begin{equation*}
\left\langle\Phi_{i}\right\rangle=\frac{1}{Z} \frac{\partial Z}{\partial J_{i}}=\frac{\partial W}{\partial J_{i}} \tag{117}
\end{equation*}
$$

The second derivative reads

$$
\begin{equation*}
\left\langle\Phi_{i} \Phi_{j}\right\rangle=\frac{1}{Z} \frac{\partial^{2} Z}{\partial J_{i} \partial J_{j}}=\frac{1}{Z} \frac{\partial}{\partial J_{i}}\left(Z \frac{\partial W}{\partial J_{j}}\right)=\left\langle\Phi_{i}\right\rangle\left\langle\Phi_{j}\right\rangle+\frac{\partial^{2} W}{\partial J_{i} \partial J_{j}} \tag{118}
\end{equation*}
$$

The first term is the disconnected part, this remains if there is no interrelation between the fields defined in $i$ and $j$, for example if they are part of independent systems.

In the quadratic model, up to a constant we have

$$
\begin{equation*}
W[J]=\ln Z_{0}+\frac{1}{2} J_{i} G_{i j} J_{j} \tag{119}
\end{equation*}
$$

where in bosonic theories from (90) $Z_{0}=\operatorname{det}^{-1 / 2} \mathcal{K}$. Using $\ln \operatorname{det} A=\operatorname{Tr} \ln A$ relation we may write

$$
\begin{equation*}
W[J]=-\frac{1}{2} \operatorname{Tr} \ln \mathcal{K}+\frac{1}{2} J_{i} G_{i j} J_{j} \tag{120}
\end{equation*}
$$

Usually we call the second derivative of the free energy $W^{(2)}$ the propagator, and we also denote it by $G$ in general.

The $W[J]$ functional in perturbation theory is related to the generator of connected diagrams. Consider the diagramatic expansion of $Z[J]$ : these are vacuum diagrams where there are connected and disconnected parts. Single out one $J_{i}$ by differentiating with respect to it. There is a part of the diagrams which is connected to $J_{i}$, but any other diagrams can appear besides it. Denoting by $W[J]$ the connected part, we can write

$$
\begin{equation*}
\frac{\partial Z[J]}{\partial J_{i}}=\frac{\partial W[J]}{\partial J_{i}} Z[J] \Rightarrow Z[J]=e^{W[J]} \tag{121}
\end{equation*}
$$

as we promised.
From the path integral definition

$$
\begin{equation*}
e^{W[J]}=\int \mathcal{D} \varphi e^{-S_{0}-S_{i n t}[\varphi]+J \varphi}=\int \mathcal{D} \varphi e^{-S_{0}+J \varphi}\left(e^{-S_{i n t}}-1+1\right)=Z_{0}\left[e^{\frac{1}{2} J G J}+\left\langle e^{-S_{i n t}}-1\right\rangle\right] \tag{122}
\end{equation*}
$$

Performing logarithm keeps only the connected parts. Taking into account that in $e^{\frac{1}{2} J G J}$ the only connected diagram is the propagator, we may write

$$
\begin{equation*}
W[J]=-\frac{1}{2} \operatorname{Tr} \ln \mathcal{K}+\frac{1}{2} J G J+\left.\left\langle e^{-S_{i n t}}-1\right\rangle\right|_{\text {connected }} \tag{123}
\end{equation*}
$$

We remark that the expansion keeping only the connected diagrams is also called cluster decomposition in statistical physics.

### 5.2 Effective action

With the Legendre transformation of the free energy we can define different thermodynamic potentials. We now perform Legendre transformation with respect to $J$ :

$$
\begin{equation*}
\Gamma[\Phi[J]]=J \Phi-W[J], \quad \Phi_{i}[J]=\frac{\partial W}{\partial J_{i}} \tag{124}
\end{equation*}
$$

In spin systems physically it means the free energy as a function of the magnetization.
We can invert the above relation and write

$$
\begin{equation*}
\Gamma[\Phi]=J \Phi-W[J], \quad J_{i}=\frac{\partial \Gamma}{\partial \Phi_{i}} \tag{125}
\end{equation*}
$$

This latter relation comes since

$$
\begin{equation*}
\frac{\partial \Gamma}{\partial \Phi_{i}}=\frac{\partial J_{k}}{\partial \Phi_{i}} \Phi_{k}+J_{i}-\frac{\partial W}{\partial J_{k}} \frac{\partial J_{k}}{\partial \Phi_{i}}=J_{i} \tag{126}
\end{equation*}
$$

If we want to compute physical quantities, the expectation valules must be evaluated at $J=0$ (cf. 64). But $J=0$ means

$$
\begin{equation*}
\left.\frac{\partial \Gamma}{\partial \Phi}\right|_{\Phi_{p h y s}}=0 \tag{127}
\end{equation*}
$$

which is the same form as the classical equation of motion. Therefore $\Gamma$ is used to call the effective action, its dentsity is the effective Lagrangian. The part of the effective Lagrangian that does not contain derivatives, ie. $\mathcal{L}_{\text {eff }}$ evaluated at constant arguments $\Phi(t, \mathbf{x})=\Phi_{0}$ is called effective potential. The coefficients in the polynomial expansion of $\Gamma$ are called proper vertices. The above equation tells us that taking into account statistical/quantum effects results in the fact that not the potential, but the effective potential is minimalized

$$
\begin{equation*}
\left.\frac{\partial U_{e f f}}{\partial \Phi}\right|_{\Phi_{0}}=0 \tag{128}
\end{equation*}
$$

Now we write up some characteristic relations between the derivatives of $W$ and $\Gamma$. Differentiating the $J$ relation from with respect to $J$ we find

$$
\begin{equation*}
\delta_{i j}=\frac{\partial^{2} \Gamma}{\partial \Phi_{i} \partial J_{j}}=\frac{\partial^{2} \Gamma}{\partial \Phi_{i} \partial \Phi_{k}} \frac{\partial \Phi_{k}}{\partial J_{i}}=\frac{\partial^{2} \Gamma}{\partial \Phi_{i} \partial \Phi_{k}} \frac{\partial^{2} W}{\partial J_{k} \partial J_{i}} \tag{129}
\end{equation*}
$$

Writing another way, using also the $G=W^{(2)}$ usual notation for the propagator

$$
\begin{equation*}
\delta_{i j}=\Gamma_{i k}^{(2)} G_{k j} \tag{130}
\end{equation*}
$$

these are inverses of each other. In the quadratic model we indeed have seen: $\mathcal{K}_{i k} G_{k j}=\delta_{i j}$. If we start from the free theory with $\mathcal{K}_{0}$ kernel and $G_{0}$ propagator, then we have

$$
\begin{equation*}
G_{0 i j}=\mathcal{K}_{0 i j}^{-1}, \quad \mathcal{K}_{i j}=\mathcal{K}_{0 i j}+\Sigma_{i j}, \quad G_{i j}=\left(\mathcal{K}_{0}+\Sigma\right)_{i j}^{-1} \tag{131}
\end{equation*}
$$

where $\Sigma$ is the self energy defined already at (114). So the self-energy is just the correction to the free kernel.
The relation of 3 -point functions is obtained by an additional derivative with respect to $J_{\ell}$ :

$$
\begin{equation*}
0=\Gamma_{i k \ell}^{(3)} G_{k j}+\Gamma_{i k}^{(2)} G_{k j m} \Gamma_{m \ell}^{(2)} \Rightarrow W_{i k \ell}^{(3)}=-G_{i i^{\prime}} G_{j j^{\prime}} G_{\ell \ell^{\prime}} \Gamma_{i^{\prime} k^{\prime} \ell^{\prime}}^{(3)} \tag{132}
\end{equation*}
$$

This is interpreted that the connected 3-point function comes from $\Gamma^{(3)}$ - the proper 3-point function - by "dressing" it with propagators. Reversely, the proper 3-point function is the amputed connected 3-point function.

With path integral we may write

$$
\begin{equation*}
e^{-\Gamma[\Phi]}=e^{-W[J]-J \Phi}=\int \mathcal{D} \varphi e^{-S[\varphi]+J(\varphi-\Phi)}=\int \mathcal{D} \varphi^{\prime} e^{-S\left[\Phi+\varphi^{\prime}\right]+J \varphi^{\prime}} \tag{133}
\end{equation*}
$$

It is the free energy with a modified action. We must not forget that $\Phi$ and $J$ are not independent. Because of their relation the one point function in this theory

$$
\begin{equation*}
\left\langle\varphi^{\prime}\right\rangle \sim \int \mathcal{D} \varphi^{\prime} \varphi^{\prime} e^{-S\left[\Phi+\varphi^{\prime}\right]+J \varphi^{\prime}}=e^{-J \Phi} \int \mathcal{D} \varphi(\varphi-\Phi) e^{-S[\varphi]+J \varphi}=0 \tag{134}
\end{equation*}
$$

So it is usual to omit the $J \varphi^{\prime}$ term and replace by the condition that $\left\langle\varphi^{\prime}\right\rangle=0$.

$$
\begin{equation*}
e^{-\Gamma[\Phi]}=\left.\int \mathcal{D} \varphi e^{-S[\Phi+\varphi]}\right|_{\langle\varphi\rangle=0} \tag{135}
\end{equation*}
$$

This is the background field method to compute $\Gamma$, where $\Phi$ is called background and $\varphi$ the fluctuation.
If we neglect fluctuations, and replace the expectation value by $\varphi=0$ - this is called mean field approximation -, then the path integral is gone and we have

$$
\begin{equation*}
\Gamma_{\text {mean field }}[\Phi]=S[\Phi] \tag{136}
\end{equation*}
$$

So $\Gamma$ is the classical action to lowest order.
In the quadratic theory

$$
\begin{equation*}
W[J]=Z_{0}+\frac{1}{2} J_{i} G_{i j} J_{j}, \quad \Phi_{i}=G_{i j} J_{j}, \quad J_{i}=\mathcal{K}_{i j} \Phi_{j} \tag{137}
\end{equation*}
$$

therefore

$$
\begin{equation*}
\Gamma[\Phi]=J_{i} \Phi_{i}-W[J[\Phi]]=\Phi_{i} \mathcal{K}_{i j} \Phi_{j}-Z_{0}-\frac{1}{2} J_{i} G_{i j} J_{j}=\frac{1}{2} \operatorname{Tr} \ln \mathcal{K}+\frac{1}{2} \Phi_{i} \mathcal{K}_{i j} \Phi_{j} \tag{138}
\end{equation*}
$$

where we put in the bosonic expression of $Z_{0}$. In the quadratic model $\Gamma=S$ : in this case, therefore, there are no fluctuations, the mean field theory is exact.

In perturbation theory $\Gamma$ is the generator of the one-particle irreducible (1PI) diagrams, diagrams that remain connected after cutting any propagator line. This is simply the consequence that in the background field method $\langle\varphi\rangle=0$, which means that all the diagrams that contain a fluctuation one-point function, are zero. From the background field method we have

$$
\begin{equation*}
e^{-\Gamma[\Phi]}=\left.\int \mathcal{D} \varphi e^{-S[\Phi]-S_{1}[\Phi, \varphi]}\right|_{1 P I}=\left.e^{-S[\Phi]} \int \mathcal{D} \varphi e^{-S_{1}[\Phi, \varphi]}\right|_{1 P I} \tag{139}
\end{equation*}
$$

where

$$
\begin{equation*}
S[\Phi+\varphi]=S[\Phi]+\frac{\partial S}{\partial \Phi_{i}} \varphi_{i}+S_{1}[\Phi, \varphi] \tag{140}
\end{equation*}
$$

the linear term is gone when we take 1PI diagrams. The $S_{1}$ part contains a quadratic and interaction part

$$
\begin{equation*}
S_{1}[\Phi, \varphi]=\frac{1}{2} \varphi_{i} S^{(2)}[\Phi]_{i j} \varphi_{j}+S_{i n t}[\Phi, \varphi] \tag{141}
\end{equation*}
$$

although it is somewhat arbitrary, which part is considered quadratic and which is interaction - we will return to this point. In perturbation theory we write

$$
\begin{equation*}
\int \mathcal{D} \varphi e^{-S_{1}[\Phi, \varphi]}=\int \mathcal{D} \varphi e^{-\frac{1}{2} \varphi S^{(2)} \varphi}+\int \mathcal{D} \varphi\left(e^{-S_{i n t}}-1\right) e^{-\frac{1}{2} \varphi S^{(2)} \varphi} \tag{142}
\end{equation*}
$$

The first term yields $Z_{0}$, calculated from $S^{(2)}[\Phi]$, in the second term we power expand $e^{-S_{i n t}}$. Finally we have to take the logarithm of the result which means that only the connected part is to be kept. We arrive at the form

$$
\begin{equation*}
\Gamma[\Phi]=S[\Phi]+\frac{1}{2} \operatorname{Tr} \ln S^{(2)}+\left\langle 1-e^{-S_{i n t}}\right\rangle_{S^{(2)}, \text { conn }, 1 P I} \tag{143}
\end{equation*}
$$

The last term means that we have to power expand the exponent, use perturbation theory to evaluate the resulting vacuum diagrams with a propagator that comes from $S^{(2)}$, and keep only the connected 1PI diagrams.

As an example we again take $\Phi^{4}$ model, and compute the effective potential. To have it we expand around a static $\Phi_{0}$ background, and use that

$$
\begin{equation*}
\Gamma\left[\Phi_{0}\right]=V U_{e f f}\left(\Phi_{0}\right) \tag{144}
\end{equation*}
$$

where $V$ is the volume of the Euclidean space. Applying partial integation we have for the shifted Lagrangian

$$
\begin{equation*}
\mathcal{L}_{E}\left(\Phi_{0}+\varphi\right)=\frac{m^{2}}{2} \Phi_{0}^{2}+\frac{\lambda}{24} \Phi_{0}^{4}+\left(m^{2}+\frac{\lambda}{6} \Phi_{0}^{2}\right) \Phi_{0} \varphi+\frac{1}{2} \varphi\left(-\partial^{2}+m^{2}+\frac{\lambda}{2} \Phi_{0}^{2}\right) \varphi+\frac{\lambda \Phi_{0}}{6} \varphi^{3}+\frac{\lambda}{24} \varphi^{4} .( \tag{145}
\end{equation*}
$$

From here we identify the terms

$$
\begin{align*}
& U=\frac{m^{2}}{2} \Phi_{0}^{2}+\frac{\lambda}{24} \Phi_{0}^{4} \\
& S^{(2)}=-\partial^{2}+m^{2}+\frac{\lambda}{2} \Phi_{0}^{2} \\
& S_{i n t}=\int d^{d} x\left[\frac{\lambda \Phi_{0}}{6} \varphi^{3}+\frac{\lambda}{24} \varphi^{4}\right] \tag{146}
\end{align*}
$$

For constant background field we have a background-dependent modified mass term, so the propagator in this case

$$
\begin{equation*}
G(p)=\frac{1}{p^{2}+M^{2}}, \quad M^{2}=m^{2}+\frac{\lambda}{2} \Phi_{0}^{2} \tag{147}
\end{equation*}
$$

The effective potential then reads

$$
\begin{equation*}
U_{e f f}=\frac{m^{2}}{2} \Phi^{2}+\frac{\lambda}{24} \Phi^{4}+\frac{1}{2} \int \frac{d^{d} p}{(2 \pi)^{d}} \ln \left(p^{2}+M\right)+U_{e f f}^{i n t} \tag{148}
\end{equation*}
$$

where to $\mathcal{O}\left(\lambda^{2}\right)$ we have

$$
\begin{equation*}
U_{e f f}^{i n t}=\frac{1}{V}\left\langle 1-e^{-S_{i n t}}\right\rangle_{S^{(2)}, c o n n, 1 P I}=\frac{1}{V}\left\langle S_{i n t}-\frac{1}{2} S_{i n t}^{2}\right\rangle_{S^{(2)}, c o n n, 1 P I}+\mathcal{O}\left(\lambda^{3}\right) \tag{149}
\end{equation*}
$$

In the expectation value of the first diagram the $\varphi^{3}$ term does not give contribution, in the expecation value of the second diagram the $\left\langle\varphi_{1}^{3} \varphi_{2}^{3}\right\rangle$ and $\left\langle\varphi_{1}^{4} \varphi_{2}^{4}\right\rangle$ expectation values have to be computed. Therefore we have, using the notation of earlier introduced $\mathcal{T}$ and $\mathcal{B}_{n}$ diagrams (cf. 105) and 111)

$$
\begin{equation*}
U_{\text {eff }}^{\text {int }}=\frac{\lambda}{8} \mathcal{T}^{2}-\frac{\lambda^{2}}{12} \Phi_{0}^{2} \mathcal{B}_{3}-\frac{\lambda^{2}}{4} \mathcal{T}^{2} \mathcal{B}_{2}-\frac{\lambda^{2}}{48} \mathcal{B}_{4} \tag{150}
\end{equation*}
$$

where $\mathcal{B}$ without argument means its value at $p=0$. The diagrams appearing in this calculation are drawn in Fig. 3.


Figure 3: Radiative corrections to the effective potential in $\Phi^{4}$ theory. Cross means background field $\Phi_{0}$.
We remark that the expansion presented here is not the strict $\lambda$ expansion, since we use a mass in the propagator that is also $\lambda$ dependent. Also we generated higher powers of $\Phi_{0}$ than in the original model. To expand the result in $\lambda$ we should use

$$
\begin{align*}
& \ln \left(p^{2}+m^{2}+\frac{\lambda}{2} \Phi_{0}^{2}\right)=\ln \left(p^{2}+m^{2}\right)+\frac{\lambda}{2} \Phi_{0}^{2} G(p)-\frac{\lambda^{2}}{8} \Phi_{0}^{4} G^{2}(p)+\ldots \\
& \left(p^{2}+m^{2}+\frac{\lambda}{2} \Phi_{0}^{2}\right)^{-1}=G(p)-\frac{\lambda}{2} \Phi_{0}^{2} G(p)+\frac{\lambda^{2}}{4} \Phi_{0}^{2} G^{3}(p)+\ldots \tag{151}
\end{align*}
$$

We find

$$
\begin{equation*}
U_{e f f}\left(\Phi_{0}=0\right)=\frac{1}{2} \int \frac{d^{d} p}{(2 \pi)^{d}} \ln \left(p^{2}+m^{2}\right)+\frac{\lambda}{8} \mathcal{T}^{2}-\frac{\lambda^{2}}{4} \mathcal{T}^{2} \mathcal{B}_{2}-\frac{\lambda^{2}}{48} \mathcal{B}_{4}+\mathcal{O}\left(\lambda^{3}\right) \tag{152}
\end{equation*}
$$

and

$$
\begin{equation*}
U_{\text {eff }}\left(\Phi_{0}\right)=U_{\text {eff }}\left(\Phi_{0}=0\right)+\frac{1}{2}\left(m^{2}+\frac{\lambda}{2} \mathcal{T}-\frac{\lambda^{2}}{4} \mathcal{T} \mathcal{B}_{2}-\frac{\lambda^{2}}{6} \mathcal{B}_{3}\right) \Phi_{0}^{2}+\frac{1}{24}\left(\lambda-\frac{3}{2} \lambda^{2} \mathcal{B}_{2}\right) \Phi_{0}^{4}+\mathcal{O}\left(\lambda^{3}\right) \tag{153}
\end{equation*}
$$

The self-energy from this expression agrees with 115 ).
We could also do this procedure with momentum dependent background field. We obtain at $\mathcal{O}\left(\lambda^{2}\right)$ :

$$
\begin{align*}
\Gamma\left[\Phi_{0}\right] & =S\left[\Phi_{0}\right]+\int \frac{d^{d} p}{(2 \pi)^{d}} \frac{1}{2} \Phi_{0}^{*}(p)\left(\frac{\lambda}{2} \mathcal{T}-\frac{\lambda^{2}}{4} \mathcal{T} \mathcal{B}_{2}(0)-\frac{\lambda^{2}}{6} \mathcal{B}_{3}(p)\right) \Phi_{0}(p)+ \\
& -\frac{1}{24} \frac{\lambda^{2}}{2} \int \prod_{a=1}^{4}\left[\frac{d^{d} p_{a}}{(2 \pi)^{d}} \Phi_{0}\left(p_{a}\right)\right](2 \pi)^{d} \delta\left(\sum_{a} p_{a}\right)\left(\mathcal{B}_{2}\left(p_{1}+p_{2}\right)+\mathcal{B}_{2}\left(p_{1}+p_{3}\right)+\mathcal{B}_{2}\left(p_{1}+p_{4}\right)\right)+ \\
& +\mathcal{O}\left(\lambda^{3}\right) \tag{154}
\end{align*}
$$

From here we can easily read off the momentum dependent second and fourth derivative of the effective action

$$
\begin{align*}
& \Gamma^{(2)}(p)=p^{2}+m^{2}+\frac{\lambda}{2} \mathcal{T}-\frac{\lambda^{2}}{4} \mathcal{T} \mathcal{B}_{2}(0)-\frac{\lambda^{2}}{6} \mathcal{B}_{3}(p)+\mathcal{O}\left(\lambda^{3}\right) \\
& \Gamma^{(4)}(p)=\lambda-\frac{1}{2} \lambda^{2}\left[\mathcal{B}_{2}\left(p_{1}+p_{2}\right)+\mathcal{B}_{2}\left(p_{1}+p_{3}\right)+\mathcal{B}_{2}\left(p_{1}+p_{4}\right)\right]+\mathcal{O}\left(\lambda^{3}\right) \tag{155}
\end{align*}
$$

## 6 Giving sense the path integral: renormalization

The path integral was defined in a discretized version, the continuum version is obtained in a limit when the regularization is vanished. If this is a sensible procedure, the terms in perturbation theory should be convergent.

Let us therefore examine the integral

$$
\begin{equation*}
\mathcal{T}_{\alpha}=\int \frac{d^{d} p}{(2 \pi)^{d}} \frac{1}{\left(p^{2}+m^{2}\right)^{\alpha}} \tag{156}
\end{equation*}
$$

The integral can be evaluated formally in any dimensions, cf. Appendix A.1 , but the general properties can be read off without the explicit result. We see that if $d \geq 2 \alpha$, then this integral is divergent. On the other hand, by dimensional reasons $\mathcal{T}_{\alpha} \sim m^{d-2 \alpha}$, which means that for $m^{2} \rightarrow 0$ it is divergent when $d<2 \alpha$.

We have seen that in $\Phi^{4}$ model at one-loop level there appears both $\mathcal{T}=\mathcal{T}_{1}$ tadpole and $\mathcal{B}_{2}=\mathcal{T}_{2}$ bubble diagrams. Therefore at $d \geq 2$ the theory is ill-defined, the tadpole integral is divergent. At $d=4$ both the tadpole and the bubble are divergent.

This is a very surprising result. So far we made everything very cautionsly, using our best knowledge of quantum mechanics and statistical physics. After a long tour de force the results are still useless?

In fact there was a long period of time, from the 1950's to 1970's when in particle physics prominent physicists thought that quantum field theory is corrupted in some unknown way, and it is better not to use it. In statistical physics the situation is better, we can always assume that there is some physical process that puts a regulator in the theory. Only in the 1970's when the theory of renormalization was worked out, was possible to restart the QFT program. This program, however, reinterprets physics itself, not just the way of performing integrals.

The first painful thing we have to go through is that we have to accept that the definition of path integral requires a regulator. This is painful, because we hoped universality in the limiting process, like in a convergent series. We can approach 0 with a lot of ways, for example like $1 / n$ or $e^{-n}$ for $n \rightarrow \infty$, but after taking the limit, we can use 0 as an entity without referring to the way we arrived at it. It seems that in case of path integral this simple convergence idea is not working.

We may ancourage ourselves thinking about distribution theory. Here we work with linear maps that map a (properly defined) function to a real number. Any function can be interpreted as a distribution by the definition

$$
\begin{equation*}
f: g \mapsto \int_{-\infty}^{\infty} d x f(x) g(x) \tag{157}
\end{equation*}
$$

provided the integral exists. Now let us consider a two-parameter function series

$$
\begin{equation*}
f_{\varepsilon, \eta}(x)=\frac{\eta}{x^{2}+\varepsilon^{2}} \tag{158}
\end{equation*}
$$

For finite $\eta$ and $\varepsilon$ this function is a well defined distribution. If $\eta \rightarrow 0$, then $\lim _{\eta \rightarrow 0} f_{\varepsilon, \eta}(x)=0$ as a function as well as a linear map. But the $\varepsilon \rightarrow 0$ limit is not well defined: as a function $\lim _{\varepsilon \rightarrow 0} f_{\varepsilon, \eta}(x)=\eta / x^{2}$ which is well defined apart from $\{x=0\}$ zero measure set. But as a distribution it is meaningless, since $g(0)$ is usually not zero, and then $g(x) / x^{2}$ is not integrable. The question is what happens when both $\eta \rightarrow 0$ and $\varepsilon \rightarrow 0$ : the result certainly depends on the way we perform the limit.

In most cases the result is trivial, the limiting function is either 0 or $1 / x^{2}$. But we can observe that the integral of $f_{\varepsilon, \eta}$ is $\pi \eta / \varepsilon$. Moreover, as $\varepsilon, \eta \rightarrow 0$, for any fixed $x$ the function behaves as $\eta / x^{2} \rightarrow 0$. So in the double zero limit the limiting function is zero almost everywhere, except $x=0$. Therefore we may write as a linear map:

$$
\begin{equation*}
\lim _{\varepsilon, \eta \rightarrow 0} f_{\varepsilon, \eta}: g \mapsto\left(\lim _{\varepsilon, \eta \rightarrow 0}\right) g(0) \tag{159}
\end{equation*}
$$

so, if the limit exists, it just evaluates its function argument at zero - so it is proportional to the Dirac-delta distribution. We see that we indeed can obtain a sensible double zero limit, if $\lim _{\varepsilon, \eta \rightarrow 0} \eta / \varepsilon=r$ is a finite constant:

$$
\begin{equation*}
\lim _{\varepsilon, \eta \rightarrow 0} f_{\varepsilon, \eta}(x)=r \delta(x) \tag{160}
\end{equation*}
$$

To control the limes, therefore, we must control $r$. This can be done by requiring that for all $\varepsilon$ and $\eta$ the constant unit function is mapped to the same number:

$$
\begin{equation*}
f_{\varepsilon, \eta}(1)=r, \quad \forall \varepsilon, \eta . \tag{161}
\end{equation*}
$$

Try to proceed in this track to better understand path integrals. Path integral is in fact a linear map that associates operators, comfortably the $n$-point correlation functions $\mathcal{A}=\Phi_{i_{1}}\left(t_{1}\right) \ldots \Phi_{i_{n}}\left(t_{n}\right)$ its expectation value. Just like $f_{\varepsilon, \eta}(x)$ above, it contains a regulator $R$, for example a spacetime mesh, or a momentum cutoff, or anything we can imagine and that gives finite result for all $n$-point correlation functions. Moreover it depends on the Euclidean action $S_{E}$. Thus it is in fact a two-parameter linear map:

$$
\begin{equation*}
\mathrm{PI}_{R, S_{E}}: \mathcal{A} \mapsto\langle\mathcal{A}\rangle_{R} \in \boldsymbol{R} \tag{162}
\end{equation*}
$$

We have seen that $\lim _{R \rightarrow 0} \mathrm{PI}_{R, S_{E}}$ does not exists, just like in the $\varepsilon \rightarrow 0$ limit in the previous example. We can also find that $\lim _{e^{-S_{E} \rightarrow 0}}^{R \rightarrow 0} \mathrm{PI}_{R, S_{E}}=0$, just like in the $\eta \rightarrow 0$ limit in the previous case. So we are in a similar situation than before. We have seen that in order to have a meaningful result we should balance the limiting procedures of the different arguments: in this case we should modify $S_{E}$ together with $R$. In the previous example the limes was governed by the ratio of the two parameters. In the path integral case we also have to fix something that governs the limit. Practically we demand that the expectation values of some singled out $\mathcal{A}_{\alpha}(\alpha=1 \ldots m)$ correlators are the same for all $R$ and $S_{E}$ :

$$
\begin{equation*}
\mathrm{PI}_{R, S_{E}}\left(\mathcal{A}_{\alpha}\right)=A_{\alpha}=\text { fixed }, \quad \forall R, \text { and } S_{E}, \quad \alpha=1 \ldots m \tag{163}
\end{equation*}
$$

This procedure is called renormalization: we can not keep the action fixed when we remove the regularization, but we can keep the observable world - the "physics" - fixed, at least some of the expectation values.

Usually we choose a set of regulators parametrized by one continuous parameter $k$ called "scale", and the removal of the regulator is at $k \rightarrow \infty$. To keep $\left\langle\mathcal{A}_{\alpha}\right\rangle$ unchanged, the action must also depend on this parameter $S_{E}(k)$. The action is built up as a sum of different operators with certain coupling constants:

$$
\begin{equation*}
S_{E}=\sum_{\ell=1}^{L} g_{\ell} O_{\ell} \tag{164}
\end{equation*}
$$

where the $O_{\ell}$ operators are for example $\Phi^{2}, \Phi^{4},\left(\partial_{\mu} \Phi\right)^{2}$ etc. Therefore $S_{E}(k)$ in fact means $g_{\ell}(k)$. In the space of all coupling constants - in this case it is $\boldsymbol{R}^{L}$ - the change of the scale corresponds to a curve. Sometimes this curve is called line of constant physics (LCP), because any theory in this curve yields with the corresponding $R(k)$ regulator the same observables (at least for the singled out $\mathcal{A}_{\alpha}$ correlators).

### 6.1 Renormalized perturbation theory

In the light we have told above let us reexamine the ideas of perturbation theory (PT). There are three elements of the renormalized perturbation theory: $R, S(R)$ and $A_{\alpha}$. The first is called regularization, the second is a form of the the renormalization scheme (to be definde later), the third is the renormalization prescription.

Originally perturbation theory provides a series in the coupling constant. But now the coupling constant is not fixed, it is changad with the scale. It is better to choose an expansion parameter that is somehow connected to physics and is unchanged with removing the regulator. Let us see how it works in $\Phi^{4}$ theory.

Let us take $\Phi^{4}$ model in $d=4$ as example. Let us choose as regulator the momentum cutoff: it is defined that we write up the path integral in momentum representation, and we integrate only over $\Phi_{p}$ with $p^{2}<\Lambda^{2}$. The regulator therefore is controled by $\Lambda$, the removal of the regulator is the $\Lambda \rightarrow \infty$ (continuum) limit. The one loop diagrams that appear in $\Gamma^{(2)}$ and $\Gamma^{(4)} 155$ read in the momentum cutoff regularization (cf. Appendix)

$$
\begin{align*}
\mathcal{T} & =\frac{1}{16 \pi^{2}}\left[\Lambda^{2}+m^{2} \ln \frac{m^{2}}{\Lambda^{2}}\right] \\
\mathcal{B}_{2}(p) & =-\frac{1}{16 \pi^{2}}\left[\ln \frac{m^{2}}{\Lambda^{2}}-1+F \ln \frac{F+1}{F-1}\right], \quad F=\sqrt{1+\frac{4 m^{2}}{p^{2}}} \tag{165}
\end{align*}
$$

In the original model there are two parameters, $m^{2}$ and $\lambda$. As we have discussed, these parameters must not kept fixed, as the regulator changes, in this case when $\Lambda$ changes, we have to adjust them in order to keep physics unchanged. To reach this goal, as it turns out, we will need at least one additional regularization dependent quantity which is implicit in the original notation. We will need to rescale the original $\Phi$ field as $\Phi=Z^{1 / 2} \varphi$, and we will compute correlation functions of $\varphi$, otherwise we can not have finite correlation function in the continuum limit. $Z$ is called wave function renormalization. The rescaled Lagrangian reads

$$
\begin{equation*}
\mathcal{L}=\frac{Z}{2}\left(\partial_{i} \varphi\right)^{2}+\frac{Z m^{2}}{2} \varphi^{2}+\frac{Z^{2} \lambda}{24} \varphi^{4} \tag{166}
\end{equation*}
$$

This form is used to call bare Lagrangian, and the parameters $m_{b}^{2}=Z m^{2}$ and $\lambda_{b}=Z^{2} \lambda^{2}$ are the bare couplings. So finally altogehter we have three parameters $m^{2}, Z$ and $\lambda$ which should be functions of $\Lambda$, and which are all go to infinity as $\Lambda \rightarrow \infty$.

The expansion parameter of the PT will be denoted by $\lambda_{\text {ren }}$, it is called renormalized coupling. In principle it can be completely independent of the original coupling $\lambda$, but in practice we assume that in the leading order $\lambda=\lambda_{\text {ren }}+\mathcal{O}\left(\lambda_{\text {ren }}^{2}\right)$. The reason is that if there were no loop corrections, then the result would be independent of $\Lambda$, therefore the cutoff dependence of the parameters of the Lagrangian start at the first nontrivial order. It is worth to remove the divergent part from the masses, too, and introduce a finite renormalized mass $m_{r e n}$. Again, in practice one assumes that $m^{2}=m_{\text {ren }}^{2}+\mathcal{O}\left(\lambda_{\text {ren }}^{2}\right)$. In principle we also could have a renormalized value for the wave function renormalization, but it is usually chosen 1 , so $Z=1+\mathcal{O}\left(\lambda_{\text {ren }}\right)$ (actually in scalar models only $\left.\mathcal{O}\left(\lambda_{\text {ren }}^{2}\right)\right)$. Thus we can write

$$
\begin{equation*}
Z=1+\delta Z, \quad Z m^{2}=m_{r e n}^{2}+\delta m^{2}, \quad Z^{2} \lambda=\lambda_{\text {ren }}+\delta \lambda \tag{167}
\end{equation*}
$$

where the correction are all power series in the expansion parameter $\lambda_{\text {ren }}$ :

$$
\begin{equation*}
\delta m^{2}=\sum_{n=1}^{\infty} \lambda_{r e n}^{n} \delta m_{n}^{2}, \quad \delta Z=\sum_{n=1}^{\infty} \lambda_{\text {ren }}^{n} \delta Z_{n}, \quad \delta \lambda=\sum_{n=1}^{\infty} \lambda_{r e n}^{n+1} \delta \lambda_{n} \tag{168}
\end{equation*}
$$

If we rewrite this expression to the Lagrangian we have

$$
\begin{equation*}
\mathcal{L}=\frac{1}{2}(\partial \varphi)^{2}+\frac{m_{r e n}^{2}}{2} \varphi^{2}+\frac{\lambda_{r e n}}{24} \varphi^{4}+\frac{\delta Z}{2}(\partial \varphi)^{2}+\frac{\delta m^{2}}{2} \varphi^{2}+\frac{\delta \lambda}{24} \varphi^{4} \tag{169}
\end{equation*}
$$

The first, quadratic part is considered the unperturbed Lagrangian. The interaction part contains the original vertex with strength $\lambda_{\text {ren }}$, and additional terms that are power series of $\lambda_{\text {ren }}$ - this is called the counterterm part. In perturbation theory we take into account that term from the counterterm part, which fits the power of $\lambda_{\text {ren }}$ of the order of the diagram. For example if we want to compute the second derivative of the effective action to one loop order then we have to consider $\delta m_{1}^{2}$, since it is of order $\lambda_{\text {ren }}$ :

$$
\begin{equation*}
\Gamma^{(2)}=p^{2}+m_{\text {ren }}^{2}+\frac{\lambda}{2} \mathcal{T}+\delta m_{1}^{2}=p^{2}+m_{\text {ren }}^{2}+\frac{\lambda_{\text {ren }}}{32 \pi^{2}}\left[\Lambda^{2}+m_{\text {ren }}^{2} \ln \frac{m_{\text {ren }}^{2}}{\Lambda^{2}}\right]+\delta m_{1}^{2} \tag{170}
\end{equation*}
$$

The role of $\delta m_{1}^{2}$ is to make $\Gamma^{(2)}$ finite, because this is an observable quantity, and all observable quantities must be finite in a consistent system. This means that the terms divergent with $\Lambda \rightarrow \infty$ must be canceled:

$$
\begin{equation*}
\delta m_{1}^{2}=\Delta m_{1}^{2}-\frac{\lambda_{r e n}}{32 \pi^{2}}\left[\Lambda^{2}+m_{r e n}^{2} \ln \frac{\mu^{2}}{\Lambda^{2}}\right] . \tag{171}
\end{equation*}
$$

Besides the divergence cancellation the counterterm may contain finite term; in fact it is not well defined, what do we consider "divergent part". The rule how we associate to the divergent expression the coresponding counterterms is called renormalization scheme. In the above expression we have split the finite part into two part, a scale parameter $\mu$ and an explicit additive term $\Delta m_{1}^{2}$. At higher loop order one usually uses the same scale parameter, while the explicit finite term may change.

In the calculation of the 4 -point function at zero mometum we need $\delta \lambda_{1}$

$$
\begin{equation*}
\Gamma^{(4)}(p=0)=\lambda_{r e n}+\frac{3 \lambda_{r e n}^{2}}{32 \pi^{2}}\left[\ln \frac{m_{r e n}^{2}}{\Lambda^{2}}+1\right]+\delta \lambda_{1} \tag{172}
\end{equation*}
$$

which implies

$$
\begin{equation*}
\delta \lambda_{1}=-\frac{3 \lambda_{r e n}^{2}}{32 \pi^{2}}\left[\ln \frac{\mu^{2}}{\Lambda^{2}}+1\right]+\Delta \lambda_{1} \tag{173}
\end{equation*}
$$

In this way we can achieve a finite result for the derivatives of the effective action

$$
\begin{align*}
& \Gamma^{(2)}=p^{2}+m_{r e n}^{2}+\frac{\lambda_{r e n} m_{r e n}^{2}}{32 \pi^{2}} \ln \frac{m_{r e n}^{2}}{\mu^{2}}+\Delta m_{1}^{2}+\mathcal{O}\left(\lambda^{2}\right) \\
& \Gamma^{(4)}(p=0)=\lambda_{r e n}+\frac{3 \lambda_{r e n}^{2}}{32 \pi^{2}} \ln \frac{m_{r e n}^{2}}{\mu^{2}}+\Delta \lambda_{1}+\mathcal{O}\left(\lambda^{3}\right) \tag{174}
\end{align*}
$$

The values of $\Delta m_{1}^{2}$ and $\Delta \lambda_{1}$ is fixed by the scheme. A simple possible choice is for example $\Delta m_{1}^{2}=0$ and $\Delta \lambda_{1}=0$ - this is the minimial subtraction scheme with cutoff regularization. Then we have

$$
\begin{align*}
& \Gamma_{\mathrm{MS}}^{(2)}=p^{2}+m_{r e n}^{2}+\frac{\lambda_{r e n} m_{r e n}^{2}}{32 \pi^{2}} \ln \frac{m_{r e n}^{2}}{\mu^{2}}, \\
& \Gamma_{\mathrm{MS}}^{(4)}(p=0)=\lambda_{r e n}+\frac{3 \lambda_{r e n}^{2}}{32 \pi^{2}} \ln \frac{m_{r e n}^{2}}{\mu^{2}} \tag{175}
\end{align*}
$$

The values of $m_{r e n}^{2}$ and $\lambda_{\text {ren }}$, as for all theoretical models, we have to deduce from observations.
This procedure makes the finite momentum part of $\Gamma^{(4)}$ also finite, since the $p$-dependent parts are finite. Indeed

$$
\begin{equation*}
\Gamma_{\mathrm{MS}}^{(4)}(p)=\lambda_{r e n}+\frac{3 \lambda_{r e n}^{2}}{32 \pi^{2}} \ln \frac{m_{r e n}^{2}}{\mu^{2}}+\Delta \lambda_{1}+\frac{\lambda_{r e n}^{2}}{32 \pi^{2}}\left[C\left(p_{1}+p_{2}\right)+C\left(p_{1}+p_{3}\right)+C\left(p_{1}+p_{2}\right)\right] \tag{176}
\end{equation*}
$$

where

$$
\begin{equation*}
C(p)=F \ln \frac{F+1}{F-1}-2, \quad F=\sqrt{1+\frac{4 m_{r e n}^{2}}{p^{2}}} \tag{177}
\end{equation*}
$$

The reason why the momentum dependent term is finite follows from the fact that the divergence is independent of the external momentum; that it is so, is simple to understand keeping the leading terms of the propagator in the $q \rightarrow \infty$ limit:

This means that in $\Gamma^{(4)}(p)-\Gamma^{(4)}(p=0)$ the divergent part cancels, so this expression is finite.
Another popular scheme is when we choose

$$
\begin{align*}
& \delta m_{1}^{2}=-\frac{\lambda_{\text {ren }}}{2} \mathcal{T}=-\frac{\lambda_{\text {ren }}}{32 \pi^{2}}\left[\Lambda^{2}+m^{2} \ln \frac{m^{2}}{\Lambda^{2}}\right] \\
& \delta \lambda_{1}=\frac{3}{2} \lambda_{\text {ren }}^{2} \mathcal{B}_{2}(p)=-\frac{3 \lambda_{\text {ren }}^{2}}{32 \pi^{2}}\left[\ln \frac{m^{2}}{\Lambda^{2}}+1\right] \tag{179}
\end{align*}
$$

which is called on-shell scheme. In this case the renormalized proper vertices read

$$
\begin{equation*}
\Gamma_{\mathrm{OMS}}^{(2)}=p^{2}+m_{r e n}^{2}, \quad \Gamma_{\mathrm{OMS}}^{(4)}(p=0)=\lambda_{r e n} \tag{180}
\end{equation*}
$$

but, of course, the 4 point vertex has a nontrivial momentum dependence.
In fact there is a theorem that claims that in $\Phi^{4}$ theory - in general in all renormalizable theories with a proper choice of the counterterms all physical observables can be made finite. In the proof one has to carefully study the source of divergences and examine the occuring divergent terms with dimensional analysis as we did before. We will not prove this theorem, but later we will see, from a more general framework, why it must be true.

But accepting that this is true, we are now in a safe situation: we can compute any physical quantities as a power series of $\lambda_{\text {ren }}$, which can be chosen small (unlike the bare coupling). Although there are some problems with the convergence of the $\lambda_{\text {ren }}$ expansion, since it turns out that the radius of convergence is zero, but still it can be treated as asymptotic series, where the first few terms already approach the exact result of the path integral in case of small coupligns. In Standard Model at high energies one can compute physical observables at 2-3 loop order ${ }^{\text {1 }}$, and we can compare the results with experimental findings. The result is a real triumph: all theoretical predictions agree with the observations within the allowed statistical errors! This proves that the method of renormalized perturbation theory indeed interprets correctly the path integral.

## 7 Changing the regularization

### 7.1 Convergence of the perturbation theory

Although we have a finite result, the convergence of the perturbative series still can be a problem. Let us consider the following problem: we want to determine the renormalized parameters of the $\Phi^{4}$ model at small momentum $p \ll m_{r e n}$. In order to achieve the best convergence can then be achived when there are no radiative corrections whatsoever: it occurs when $\mu=m_{\text {ren }}$. In this case, namely

$$
\begin{equation*}
\Gamma^{(2)}=p^{2}+m_{r e n}^{2}, \quad \Gamma^{(4)}(p=0)=\lambda_{r e n} \tag{181}
\end{equation*}
$$

But this does not mean that all radiative corrections are zero. Let us compute the 4 -point function with momenta $P=\left\{p_{1}=p_{2}=p / 2, p_{3}=p_{4}=-p / 2\right\}$. Then, using $C(0)=0$ :

$$
\begin{equation*}
\Gamma^{(4)}(P)=\lambda_{r e n}+\frac{\lambda_{r e n}^{2}}{32 \pi^{2}} C(p) \tag{182}
\end{equation*}
$$

For example if $p \gg m_{r e n}^{2}$ we obtain

$$
\begin{equation*}
\Gamma^{(4)}(P)=\lambda_{r e n}+\frac{\lambda_{r e n}^{2}}{32 \pi^{2}}\left[\ln \frac{p^{2}}{m_{r e n}^{2}}-2\right] . \tag{183}
\end{equation*}
$$

Perturbation theory is reliable if the correction os smaller than the unperturbed result. This means that the PT for $\Gamma^{(4)}$ is reliable only for

$$
\begin{equation*}
p<m_{r e n} e^{\frac{16 \pi^{2}}{\lambda_{r e n}}} . \tag{184}
\end{equation*}
$$

For small $\lambda_{\text {ren }}$ the exponent is very large, even for $\lambda_{\text {ren }}=1$ the value of the exponent is $\sim 10^{68}$, in this case we do not have to worry about the validity range. On the other hand, $\lambda_{\text {ren }}$ can easily be turn out to be of order 100 , where the exponent is $(4.85)^{100 / \lambda_{r e n}}$. So this constraint can easily be severe for somewhat strongly coupled systems.

Therefore, although with the choice of $\mu^{2}$ we wanted to improve the convergence of the perturbative series, we see that we can not do it for all momenta. If we improve PT at small momenta, at high momenta we get into trouble, and if we improve PT at large momenta, we are in trouble at small momenta. The interesting is that there is always a way to improve PT, but it is different under different external circumstances.

There is a similar problem in statistical physics. There the external mass is temperature dependent, usually one chooses $m_{r e n}^{2}=C\left(T-T_{c}\right)$. At low temperature we are in the SSB phase, and the physical mass is $m_{\text {phys }}^{2}=-2 m_{r e n}^{2}=2 C T_{c}$, cf. 75 . If we want to fix the values of the couplings at low temperature, we should use $\mu^{2}=2 C T_{c}$; but then at higher temperatures we have

$$
\begin{equation*}
\Gamma_{\mathrm{MS}}^{(2)}(p=0)=C T_{c} t\left[1+\frac{\lambda_{r e n}}{32 \pi^{2}} \ln \frac{t}{2}\right], \quad t=\frac{T-T_{c}}{T_{c}} \tag{185}
\end{equation*}
$$

$t$ is the reduced temperature. If the reduced temperature goes to zero, the radiative corrections diverge. PT is reliable only for

$$
\begin{equation*}
t>2 e^{-\frac{32 \pi^{2}}{\lambda_{r e n}}} \tag{186}
\end{equation*}
$$

[^0]For $\lambda_{\text {ren }}=100$ this means $t>0.08$, ie. we cannot approach the critical temperature closer than $8 \%$.
The best would be, of course, if we could use always different renormalization where the perturbation theory converges the best. This does not mean, of course that we could simply change $\mu$ in the different situations, since it changes the regulator, and we have learned that to keep physics constant, we should also change $S_{E}$. So let us study in more detail, how the action changes when we change regularization.

### 7.2 The regularization dependent action

So let us examine, how shall we change the action in order to have the same physics. We will consider the regulator as part of the action:

$$
\begin{equation*}
Z[J]=\int \mathcal{D} \varphi e^{-S_{\text {reg }}[\varphi]+J \varphi} \tag{187}
\end{equation*}
$$

Momentum cutoff corresponds for example to adding to the original action a term:

$$
S_{\text {reg }}=S_{E}+S_{\Lambda}, \quad S_{\Lambda}=\int \frac{d^{d} p}{(2 \pi)^{d}} \frac{R_{\Lambda}(p)}{2}|\Phi(p)|^{2}, \quad R_{\Lambda}(p)= \begin{cases}0, & \text { for }|p| \leq \Lambda  \tag{188}\\ \infty, & \text { for }|p|>\Lambda\end{cases}
$$

In this way the propagator for $|p|>\Lambda$ is zero, thus making all diagrams UV finite.
Lattice regularization can also be described in this way. Consider a rectangular lattice with lattice spacings $a_{i}$ in the $i$ th direction. Then we allow only positions $x=\sum_{i} n_{i} a_{i} e_{i}$, where $n_{i} \in \boldsymbol{Z}$ integer number and $e_{i}$ is the unit vector pointing to the $i$ th direction. Partial derivation is represented as

$$
\begin{equation*}
\partial_{i} \Phi(x) \rightarrow \frac{\Phi\left(x+a_{i} e_{i}\right)-\Phi(x)}{a_{i}} \tag{189}
\end{equation*}
$$

In Fourier space the wave vectors can be only in $k_{i} \in\left[-\pi / a_{i}, \pi / a_{i}\right]$ (in a generic lattice we are in the first Brillouin zone), and the fields are periodic functions. The derivative in the Fourier representation reads

$$
\begin{equation*}
\partial_{i} \rightarrow \frac{e^{-i p_{i} a_{i}}-1}{a_{i}} \tag{190}
\end{equation*}
$$

in the $a_{i} \rightarrow 0$ limit it indeed goes to $-i p_{i}$. Then the derivative part of the action reads

$$
\begin{equation*}
\int d^{d} x \sum_{i}\left(\partial_{i} \Phi(x)\right)^{2}=\int_{\mathcal{B}} \frac{d^{d} p}{(2 \pi)^{d}} \Phi^{*}(p) \Phi(p) \sum_{i} \frac{4}{a_{i}^{2}} \sin ^{2} \frac{p_{i} a_{i}}{2} \tag{191}
\end{equation*}
$$

where $\mathcal{B}$ denotes the Brillouin zone.
Lattice regularization therefore means that we integrate always over the first Brillouin zone - this corresponds to a momentum cutoff - , and in addition in the kinetic part, instead of the original $p^{2}$, we use the above expression.

These choices are not exceptional, we may have any other one, provided the propagator in the regulated theory decreases at least as $1 / p^{d+\varepsilon}$ with $\varepsilon>0$. To be able to use analytic methods we assume that the regulator affects only the quadratic part. In matrix notation

$$
\begin{equation*}
S_{r e g}=S_{2 r e g}+S_{i n t}, \quad S_{2 r e g}=\frac{1}{2} \Phi_{i} K_{k, i j} \Phi_{j} \equiv \frac{1}{2} \Phi K_{k} \Phi \tag{192}
\end{equation*}
$$

We will restrict the generality somewhat and use a one-parameter family of regulators, parametrized with the value of $k$, the "scale". If we change $k$, the path integral is changed, in the present representation it is due to the modified propagator.

Our goal is to keep physics unchanged, this can be achieved by choosing a scale-dependent action. Let us assume that we start with a theory with $k_{0}$ scale, and try to construct that theory that is equivalent with the original one, but uses a regulator scale $k$. The key for the analytic treatment is to split the propagator at $k_{0}$, denoted by $G(p)$ into two parts, an "UV" part where typically the modulus of the momenta are larger than $k$ (denoted by $\left.G_{>k}(p)\right)$ and "IR" part where the momenta are smaller then $k$ (denoted by $G_{<k}(p)$ ):

$$
\begin{equation*}
G(p)=G_{>k}(p)+G_{<k}(p) \tag{193}
\end{equation*}
$$

We will construct the IR part that it should have no direct dependence on the high scale modes. This can be assured if the propagation of the $|p|>k$ modes are fully described by $G_{>k}(p)$, put in another way $G_{>k}(p)=G(p)$ for $p>k$.

To see some examples: with momentum cutoff we divide the propagator as

$$
\begin{equation*}
\frac{1}{p^{2}+m^{2}}=\frac{\Theta\left(p^{2}-k^{2}\right)}{p^{2}+m^{2}}+\frac{\Theta\left(k^{2}-p^{2}\right)}{p^{2}+m^{2}} \tag{194}
\end{equation*}
$$

This means that

$$
K_{>k}(p)=\left\{\begin{array}{cl}
p^{2}+m^{2}, & \text { if } p>k  \tag{195}\\
\infty, & \text { if } p<k
\end{array}\right\}, \quad K_{<k}(p)=\left\{\begin{array}{cl}
\infty, & \text { if } p>k \\
p^{2}+m^{2}, & \text { if } p<k
\end{array}\right\}
$$

Another example which is often used in the functional renormalization group (FRG) applications is Litim's regulator, corresponding to

$$
\begin{equation*}
K_{>k}(p)=p^{2} \Theta(p-k)+k^{2} \Theta(k-p)+m^{2} \tag{196}
\end{equation*}
$$

and so

$$
\begin{equation*}
G_{>k}(p)=\frac{1}{K_{>k}(p)}=\frac{\Theta(p-k)}{p^{2}+m^{2}}-\frac{\Theta(k-p)}{k^{2}+m^{2}} \tag{197}
\end{equation*}
$$

which means

$$
\begin{equation*}
G_{<k}(p)=\frac{1}{K_{<k}(p)}=\Theta(k-p)\left[\frac{1}{p^{2}+m^{2}}-\frac{1}{k^{2}+m^{2}}\right] \tag{198}
\end{equation*}
$$

thus

$$
\begin{equation*}
K_{<k}(p)=\frac{\left(p^{2}+m^{2}\right)\left(k^{2}+m^{2}\right)}{k^{2}-p^{2}} \tag{199}
\end{equation*}
$$

This diverges at $p=k$, similarly to the cutoff case, but it is a continuous function below that scale.
Once we have split the propagator, we represent the two pieces with an action containing two fields, $\varphi$ for the UV fields and $\Phi$ for the IR fields. The quadratic part of the original Lagrangian is therefore represented now as

$$
\begin{equation*}
S_{2}=\frac{1}{2}\left[\Phi K_{<k} \Phi+\varphi K_{>k} \varphi\right] \tag{200}
\end{equation*}
$$

In all diagrams we have to use the complete propagator, this is $\langle\Phi \Phi\rangle+\langle\varphi \varphi\rangle$, which can equivalently written as $\langle(\Phi+\varphi)(\Phi+\varphi)\rangle$, since there is no free cross terms between the two fields. This means that to generate the original diagrams we should use $S_{\text {int }}(\Phi+\varphi)$ in the original path integral.

Therefore the original generating functional can be written

$$
\begin{equation*}
Z[J]=\int \mathcal{D} \Phi \mathcal{D} \varphi e^{-\frac{1}{2}\left[\Phi K_{<k} \Phi+\varphi K_{>k} \varphi\right]-S_{\text {int }}(\Phi+\varphi)+J(\Phi+\varphi)} \tag{201}
\end{equation*}
$$

The two integrations can be performed subsequently

$$
\begin{equation*}
Z[J]=\int \mathcal{D} \Phi e^{-\frac{1}{2} \Phi K_{<k} \Phi+J \Phi} \int \mathcal{D} \varphi e^{-\frac{1}{2} \varphi K_{>k} \varphi-S_{\text {int }}(\Phi+\varphi)+J \varphi} \tag{202}
\end{equation*}
$$

If we compute correlation functions, then it is partly comes from the $\Phi$ correlation functions, and partly from $\varphi$ correlation function. But after performing the $\varphi$ path integral, the $\varphi$ correlation functions become function(al)s of $\Phi$, thus again lead to correlation functions of the IR modes. So after all we can reconstruct all original correlation functions from $\Phi$ correlation functions alone. This means that all information is contained in the integral where the $J \varphi$ term is missing.

We can then write the generator functional as

$$
\begin{equation*}
Z[J]=\int \mathcal{D} \Phi e^{-S_{<, k}[\Phi]+J \Phi} \tag{203}
\end{equation*}
$$

where

$$
\begin{equation*}
e^{-S_{<, k}[\Phi]}=e^{-\frac{1}{2} \Phi K_{<k} \Phi} \int \mathcal{D} \varphi e^{-\frac{1}{2} \varphi K_{>k} \varphi-S_{i n t}(\varphi+\Phi)} . \tag{204}
\end{equation*}
$$

In this way we reached our goal: we have found a representation of the action where only the IR modes are present, and we can construct the $k$-dependent effective IR action, too.

The expression of the IR action $S_{<, k}$ is very similar to the effective action obtained from the high scale modes (cf. 139 ), but here also the 1PR diagrams should be included. However, with a good scale separation, for a 1PR diagram we would require that the sum of momenta coming from the IR modes are added to an UV momentum. This has a very small phase space, and thus one usually neglects it, so we will compute

$$
\begin{equation*}
e^{-S_{<, k}[\Phi]}=\left.e^{-\frac{1}{2} \Phi\left(K_{<k}-K_{>k}\right) \Phi(p)} \int \mathcal{D} \varphi e^{-S_{>}(\varphi+\Phi)}\right|_{1 P I} \tag{205}
\end{equation*}
$$

here the last term is the effective action coming from the classical action

$$
\begin{equation*}
S_{>}(\varphi)=\frac{1}{2} \varphi K_{>k} \varphi+S_{i n t}(\varphi) \tag{206}
\end{equation*}
$$

To proceed we write the kernels $K_{>}$and $K_{<}$as the oringinal kernel plus a regulator term

$$
\begin{equation*}
K_{>k}(p)=K_{0}(p)+R_{k}(p), \quad K_{<k}(p)=K_{0}(p)+\tilde{R}_{k}(p) \tag{207}
\end{equation*}
$$

where the UV regulator $R_{k}$ should satisfy

$$
R_{k}(p)=\left\{\begin{array}{ll}
0, & \text { for } k \rightarrow 0  \tag{208}\\
\infty, & \text { for } k \rightarrow \infty,
\end{array} \quad R_{k}(p)= \begin{cases}0, & \text { for } p>k \\
\text { arbitrary }, & \text { for } p<k\end{cases}\right.
$$

although practically at $k \rightarrow \Lambda$ the regulator goes to a large value, eg. $\Lambda^{2}$.
Now we can write

$$
\begin{equation*}
S_{<}(\varphi)=\Gamma_{>}(\varphi)+\frac{1}{2} \varphi \tilde{R}_{k} \varphi, \quad S_{>}(\varphi)=S_{E}(\varphi)+\frac{1}{2} \varphi R_{k} \varphi \tag{209}
\end{equation*}
$$

and the effective action is

$$
\begin{equation*}
e^{-\Gamma>[\Phi]}=\left.e^{\frac{1}{2} \Phi R_{k} \Phi} \int \mathcal{D} \varphi e^{-S_{>}(\varphi+\Phi)}\right|_{1 P I} \tag{210}
\end{equation*}
$$

If we denote the generator functional of the UV modes as $W_{>}[J]$, then

$$
\begin{gather*}
e^{-\Gamma_{>}[\Phi]}=e^{\frac{1}{2} \Phi R_{k} \Phi} e^{W[J]-J \Phi},  \tag{211}\\
\Gamma_{>}[\Phi]=J \Phi-\frac{1}{2} \Phi R_{k} \Phi-W_{>}[J] . \tag{212}
\end{gather*}
$$

As we see, in addition to the usual Legendre transformation we have to subtract the regulator, too.

### 7.3 Exact evolution equation for $\Gamma_{>}$

To determine the exact equation for the scale dependence of the quantum effective action we compute $\Gamma_{>}$ from the generating functional

$$
\begin{equation*}
Z[J]=e^{W[J]}=\int \mathcal{D} \varphi e^{-S_{E}(\varphi)-\frac{1}{2} \varphi R_{k} \varphi+J \varphi} \tag{213}
\end{equation*}
$$

$Z[J]$ is $k$-dependent, but this dependence is quite explicit, which makes it possible to determine the derivative; in matrix notation

$$
\begin{equation*}
\partial_{k} Z=-\frac{1}{2} \partial_{k} R_{k, i j} \int \mathcal{D} \varphi \varphi_{i} \varphi_{j} e^{-S(\varphi)-\frac{1}{2} \varphi R \varphi+J \varphi} \tag{214}
\end{equation*}
$$

The right hand side, on the other hand, can be computed from the $J$-derivatives, too

$$
\begin{equation*}
\partial_{k} Z=-\frac{1}{2} \partial_{k} R_{k, i j} \frac{\partial^{2} Z}{\partial J_{i} \partial J_{j}} . \tag{215}
\end{equation*}
$$

This is an exact (functional) partial differential equation for the partition function!
We can write up the corresponding differential equations for $W$ and finally for $\Gamma_{>}$. Using

$$
\begin{equation*}
\partial_{k} e^{W}=e^{W} \partial_{k} W, \quad \frac{\partial^{2} e^{W}}{\partial J_{i} \partial J_{j}}=\frac{\partial}{\partial J_{j}}\left(\frac{\partial W}{\partial J_{i}} e^{W}\right)=e^{W}\left(\frac{\partial W}{\partial J_{i}} \frac{\partial W}{\partial J_{i}}+\frac{\partial^{2} W}{\partial J_{i} \partial J_{j}}\right) \tag{216}
\end{equation*}
$$

we find

$$
\begin{equation*}
\partial_{k} W=-\frac{1}{2} \partial_{k} R_{k, i j}\left(\frac{\partial W}{\partial J_{i}} \frac{\partial W}{\partial J_{j}}+\frac{\partial^{2} W}{\partial J_{i} \partial J_{j}}\right) \tag{217}
\end{equation*}
$$

Then we can write

$$
\begin{equation*}
\frac{\partial \Gamma_{>}}{\partial \Phi_{i}}=-R_{i j} \Phi_{j}+J_{i}+\Phi_{j} \frac{\partial J_{j}}{\partial \Phi_{i}}-\frac{\partial W}{\partial J_{j}} \frac{\partial J_{j}}{\partial \Phi_{i}}=-R_{i j} \Phi_{j}+J_{i} \tag{218}
\end{equation*}
$$

because $\Phi_{i}=\partial W / \partial J_{i}$. For the second derivative

$$
\begin{equation*}
\frac{\partial^{2} \Gamma_{>}}{\partial \Phi_{i} \partial \Phi_{j}}+R_{i j}=\left(\frac{\partial \Phi_{i}}{\partial J_{j}}\right)^{-1}=\left(\frac{\partial^{2} W}{\partial J_{i} \partial J_{j}}\right)^{-1} \equiv G_{r e g, i j}^{-1} \tag{219}
\end{equation*}
$$

where the last term is a notation.
For the $k$ derivative we have to take into account that the $\Phi-J$ relation is also $k$-dependent. So we have

$$
\begin{equation*}
\frac{\partial \Gamma_{>}[\Phi[J]]}{\partial k}=\partial_{k} \Gamma_{>}+\frac{\partial \Gamma_{>}}{\partial \Phi} \partial_{k} \Phi=\partial_{k} \Gamma_{>}-\partial_{k} \Phi R_{k} \Phi+J \partial_{k} \Phi \tag{220}
\end{equation*}
$$

On the other hand we can use the relation of $\Gamma_{>}$and $W$ :

$$
\begin{align*}
& \frac{d \Gamma_{>}[\Phi[J]]}{d k}=-\left(\partial_{k} \Phi\right) R_{k} \Phi-\frac{1}{2} \Phi \partial_{k} R_{k} \Phi+J \partial_{k} \Phi-\partial_{k} W[J]= \\
& =-\left(\partial_{k} \Phi\right) R_{k} \Phi-\frac{1}{2} \Phi \partial_{k} R_{k} \Phi+J \partial_{k} \Phi+\frac{1}{2} \partial_{k} R_{k, i j}\left(\Phi_{i} \Phi_{j}+G_{r e g, i j}\right)= \\
& =-\left(\partial_{k} \Phi\right) R_{k} \Phi+J \partial_{k} \Phi+\frac{1}{2} \operatorname{Tr} \partial_{k} R_{k} G_{r e g} \tag{221}
\end{align*}
$$

Comparing the two results we find

$$
\begin{equation*}
\partial_{k} \Gamma_{>}=\frac{1}{2} \operatorname{Tr} \partial_{k} R_{k} G_{r e g}, \quad G_{r e g}^{-1}=\frac{\partial^{2} \Gamma_{>}}{\partial \Phi \partial \Phi}+R_{k} \tag{222}
\end{equation*}
$$

This is the functional renormalization group equation (FRG), or Wetterich equation. Sometimes it is referred as the exact renormalization group (ERG) equation, an alternative name for the $\Gamma_{>}$exact effective action is perfect action. The FRG equation was written down by Ch. Wetterich and T.R. Morris in 1993, but similar equations are derived by several people earlier: Wegner, Houghton, Polchinsky, and a numerical definition was given by P. Hasenfratz and F. Niedermayer.

An alternative form can also be well used:

$$
\begin{equation*}
\partial_{k} \Gamma_{>}=\frac{1}{2} \hat{\partial}_{k} \operatorname{Tr} \ln \left(\Gamma_{>}^{(2)}+R\right) \tag{223}
\end{equation*}
$$

where the meaning of $\hat{\partial}_{k}$ is that it acts only the $k$-dependence of the regulator, treating the second derivative of the effective action as $k$-independent quantity.

## Remarks

1. Formally the $\log$ expression is a one-loop result, cf. (143), where the regularization dependence first appears. Technically it means that we can simply compute the one loop expression, and take its derivative in a way, that only the regulator is changed. Philosophically it means that if we consider all the possible interactions, then a one-loop expression is exact!
2. The above line of thought can be repeated without assuming that the regulators come from a oneparameter family. Starting from a quadratic regulator and then changing it by $R \rightarrow R+d R$, the change in $Z$ remains a 2-point function, as before. It can also be rewritten to the effective action and find

$$
\begin{equation*}
d \Gamma_{>}=\frac{1}{2} \operatorname{Tr} d R G_{r e g}, \quad G_{r e g}^{-1}=\frac{\partial^{2} \Gamma_{>}}{\partial \Phi \partial \Phi}+R \tag{224}
\end{equation*}
$$

or

$$
\begin{equation*}
d \Gamma_{>}=\frac{1}{2} d_{R} \operatorname{Tr} \ln \left(\Gamma_{>}^{(2)}+R\right) \tag{225}
\end{equation*}
$$

In this case the solution is $\Gamma_{>}[R, \Phi]$ a double functional over the regulators and the configurations. We could be even more general, and use not just a quadratic regulator, but then the whole formalism becomes very tedious and complicated.

### 7.4 Technical addon: the second derivative of the FRG equation

In this technical subsection we compute the second derivative of the FRG equation. The left hand side is easy, it is just $\partial_{k} \Gamma_{>, a b}^{(2)}$. On the right hand side

$$
\begin{equation*}
\frac{\partial}{\partial \Phi_{a}} \operatorname{Tr} \ln \left(\Gamma_{>}^{(2)}+R\right)=\Gamma_{>, a c d}^{(3)} G_{c d}, \quad G^{-1}=\Gamma_{>}^{(2)}+R \tag{226}
\end{equation*}
$$

Another derivative reads

$$
\begin{equation*}
\frac{\partial^{2}}{\partial \Phi_{a} \partial \Phi_{b}} \operatorname{Tr} \ln \left(\Gamma_{>}^{(2)}+R\right)=\Gamma_{>, a b c d}^{(4)} G_{d c}-\Gamma_{>, a c d}^{(3)} G_{d e} \Gamma_{>, b e f}^{(3)} G_{f c} \tag{227}
\end{equation*}
$$

Therefore we can write

$$
\begin{equation*}
\partial_{k} \Gamma_{>, a b}^{(2)}=\frac{1}{2} \hat{\partial}_{k}\left[\Gamma_{>, a b c d}^{(4)} G_{d c}-\Gamma_{>, a c d}^{(3)} G_{d e} \Gamma_{>, b e f}^{(3)} G_{f c}\right] \tag{228}
\end{equation*}
$$

Formally it is again a one-loop expression, as is shown at Fig. 4.


Figure 4: Graphical representation of the second derivative of right hand side of the FRG equation.

### 7.5 Interpretation of the FRG equation

The first use of the FRG equation is to derive the quantum effective action for the IR theory, discussed in the previous subsections. It is worth to emphasize once again, that using only IR degrees of freedom we do not necessarily lose information, but eventually the calculation of the original correlation functions can be very tedious. We will return to this point later.

But the quantum effective action can be interpreted in a different way, too. This leads to the realization of a measurement in quantum field theory. Let us think it through, what happens when a high energy experiment is performed, for example in the LHC accelerator in CERN. The size of the accelerator itself, and the detectors, too, are enormous, these are very far from the scale of the microphysics. But in a very clever and thorough way this gigantic machine was intentionally made insensitive to all the physics on scales larger than, say $10^{-20} \mathrm{~m}$. It is not an easy task, but, actually, every experiment that wants to study the matter at a certain resolution and magnification works in the same way. Mathematically speaking one
concrete experimental measurement that resolves the matter at scale $k$ can be described by an operator $\mathcal{M}_{k}$. The result of the experiment is its expectation value, in the path integral representation

$$
\begin{equation*}
\left\langle\mathcal{M}_{k}\right\rangle=\frac{1}{Z} \int \mathcal{D} \Phi \mathcal{M}_{k}[\Phi] e^{i S} \tag{229}
\end{equation*}
$$

As we have said, the experiment was designed to be insensitive to all disturbances coming from the $p<k$ regime: this means that $\mathcal{M}_{k} \sim e^{-S_{r e g, k}}$, where $S_{r e g, k}$ becomes large for small momentum modes. This is just the UV theory that contains an IR regulator. This means that in order to describe the measurements in the experiment with scale $k$ we should compute the correlation functions with the action $S_{>}$! In this case, therefore, the regulator is physical, it follows from the architecture and design of the measurement device.

This means that in fact $\Gamma_{>}$is physically measurable with a device that is insensitive to scales smaller than $k$. The actual form of the regulator corresponds to the actual manifestation of the hardware of the measurement device, so it is not universal. The question, why do we still have a coherent picture about the physics at, say, quark scale, will be posed later.

We can also realize that at $k=0$ the UV regulator vanishes, and we get back the full path integral. This is the third interpretation, or use of the $\Gamma_{>}$quantum effective action: it interpolates with the help of a partial differential equation between the UV scale and IR scale. When $k=0$, then the regulator is no more present in the theory (except the one we imposed at $k_{0}$ starting scale), and so the result is independent on the actual choice of the regulators at intermediate scales.

It is a sensitive question, what should be the starting point of the evolution of the FRG equation. Usually one starts from the classical action, with only the usual renormalizable operators present. But, as a matter of fact, we should know the complete physics that certain scale in order to know it in another scale. It seems to be very hard, since we can measure only those observables that have a significant value, while unimportant effects are impossible to measure: partly because their value is small, partly because there are infinitely many "not important" quantities. This means that the exact action, in all of the details is not possible to know at any scale.

### 7.6 Characterization of the effective action

Now let us turn to technicalities, how we can solve the FRG equation 223). It is a functional differential equation, so the solution is $\Gamma_{>}[k ; \Phi]$ is a function of $k$ and functional of $\Phi$, the configurations.

In general if we have a function $f: A \rightarrow B$ where $A$ and $B$ are two finite sets, then the dimension of the set of all the maps $F=\{f \mid f: A \rightarrow B\}$ is $D_{F}=D_{B}^{D_{A}}$ (power set ${ }^{2}$. If $f: A \rightarrow \mathbf{R}$ then its cardinality is $\mathbf{R}^{\mathbf{D}_{\mathrm{A}}}$, and it can be most comfortably characterized by $D_{A}$ real numbers $f_{A} \in \mathbf{R}$. To characterize a function $f: \mathbf{R} \rightarrow \mathbf{R}$ is more tedious, since there the domain set is (uncountably) infinite. Here one usually characterizes a simpler problem, for example when $f$ is complex holomorphic (analytic) and usually infinitely differentiable (ie. $F=C_{\infty}$ ). In this case it can be expanded in Taylor series that is well behaved at least in the domain of convergence. For a better description can be given with Laurent series (or Padé approximants) that is convergent in the whole analytic domain. Non-analytic functions are very hard to characterize, and in physics usually we pretend that all functions are analytic in the whole domain. But in this case $F$ is a countably infinite set that can be characterized by the values of the Laurent coefficients.

If we are working on a finite mesh as the domain of the configurations, then $\Phi: A \rightarrow \mathbf{R}$ (or any other finite dimensional vector space), where $A$ is a finite set. Therefore it can be characterized pointwise as $\Phi_{A} \in \mathbf{R}$. Therefore $\Gamma: \mathbf{R}^{\mathbf{N}} \rightarrow \mathbf{R}$ where $N$ is the dimension of the spacetime lattice. Now assuming that $\Gamma\left(\Phi_{1}, \Phi_{2}, \ldots \Phi_{N}\right)$ is analytic in all variables, then the set of all actions $H=\{\Gamma\}$ is also a countably infinite set. We could characterize it with a series

$$
\begin{equation*}
\Gamma\left(\Phi_{1}, \Phi_{2}, \ldots \Phi_{N}\right)=\sum_{n_{1}, \ldots n_{N}=-\infty}^{\infty} \tau_{n_{1}, \ldots n_{N}} \Phi_{1}^{n_{1}} \ldots \Phi_{N}^{n_{N}} \tag{230}
\end{equation*}
$$

In this case therefore there exists a countable set, the operator basis $\tilde{O}_{\{n\}}=\Phi_{1}^{n_{1}} \ldots \Phi_{N}^{n_{N}}$ that is appropriate with linear combinations to describe any action.

[^1]Although it is a valid characterization, usually one uses a different operators basis. In most cases, namely, the most important configurations are those that are close to a homogeneous one: if $\Phi_{1}=\cdots=\Phi_{N}=\Phi$ then $\Gamma(\Phi, \ldots, \Phi)=V U(\Phi)$, where $V$ is the volume and $U$ is the effective potential. Then we can define derivatives in a discrete manner, for example

$$
\begin{equation*}
\partial \Phi_{i}=\Phi_{i+1}-\Phi_{i}, \quad \partial^{2} \Phi_{i}=\partial \Phi_{i+1}-\partial \Phi_{i}, \quad \ldots \quad \partial^{n+1} \Phi_{i}=\partial^{n} \Phi_{i+1}-\partial^{n} \Phi_{i} \tag{231}
\end{equation*}
$$

Then using the condition of homogeneity in spacetime we write the action as

$$
\begin{equation*}
\Gamma[\Phi]=\sum_{x} \mathcal{L}\left(\Phi_{x}, \partial \Phi_{x}, \ldots, \partial^{N-1} \Phi_{x}\right) \tag{232}
\end{equation*}
$$

and we power expand the Lagrangian density in their variables. Usually one assumes Taylor expansibility (sic!) and write

$$
\begin{equation*}
\mathcal{L}\left(\Phi, \partial \Phi, \ldots \partial^{N-1} \Phi\right)=\sum_{n_{0}, \ldots n_{N-1}=0}^{\infty} g_{n_{0}, \ldots n_{N-1}} \Phi^{n_{0}}(\partial \Phi)^{n_{1}} \ldots\left(\partial^{N-1} \Phi\right)^{n_{N-1}} \tag{233}
\end{equation*}
$$

One can relate the coefficients $\tau_{\{n\}}$ and $g_{\{n\}}$, so the two descriptions are equivalent. Usually the gradient expanded form is used to characterize the effective action:

$$
\begin{equation*}
O_{\{n\}}=\Phi^{n_{0}}(\partial \Phi)^{n_{1}} \ldots\left(\partial^{N-1} \Phi\right)^{n_{N-1}} \tag{234}
\end{equation*}
$$

It must be evident, however, that any operator bases are equivalent, provided the coefficients can be expressed from one to each other.

If we characterized the action, then we can also describe scale dependence in this form. Since the basis forms a countable set, we can index the basis with natural numbers, and write

$$
\begin{equation*}
\mathcal{L}_{>}(k ; \Phi)=\sum_{n} g_{n}(k) O_{n} . \tag{235}
\end{equation*}
$$

The coefficients of the Lagrangian density are called (generalized) coupling constants, and their $k$ dependent form is usually referred to as running coupling.

All this means that the exact scale dependence can be described as a curve in the infinite dimensional space of the coeffients of the operator basis. This generalizes the notion of line of constant physics (LCP) defined earlier. As we have seen we have ensured with mathematically correct derivation that the physics is indeed the same for all the scales, using $\Gamma[k, \Phi]$ as the quantum effective action with a regulator term $\tilde{R}$, as it was described in 203, 209) and (223).

A very important aspect of the exact renormalization process that there is no information loss! We may choose any scale we want, using the correct effective action and the correct relation of the correlation functions we can calculate any correlations functions! We can think about the quantum actions at different scales as different characterization of the same reality, the same physics.

We can obtain evolution equation for the coupling constants if we expand both sides of the FRG equation 223) according to the operator basis and match the resulting coefficients. We write it as

$$
\begin{equation*}
k \frac{d g_{n}}{d k}=\beta_{n}(k, g) \tag{236}
\end{equation*}
$$

where the right hand side follows from the right hand side of 223 .
We can also to introduce dimensionless couplings by rescaling with the corresponding engineering dimension, $g_{n}=k^{d_{n}} h_{n}$, and also we use the logarithm of $k$ as a variable $t=\ln \frac{k}{k_{0}}$. Then the only dimensionful quantity remains $k$, and so it can not show up in the equation. We then have

$$
\begin{equation*}
\partial_{t} h_{n}=-d_{n} h_{n}+\beta_{n}(h) \tag{237}
\end{equation*}
$$

The right hand side is usually called beta-functions.

Sometimes these equations are called renormalization group (RG) equation. It is a group in that sense that the coefficient set at scales $k_{0}$ and $k$ are related, there is a transformation that maps $g_{\{n\}}\left(k_{0}\right)$ and $g_{\{n\}}(k)$. This forms a one-parameter set of transformations that has a unit element at $k=k_{0}$, invertable and performing two transformations subsequently can be written as single transformation. So these transformations form a (one-parameter) Lie-group.

In practice, unfortunately, we can not treat an infinite basis, infinite number of coefficients. We must choose a finite subset, and solve the (237) equations in this subset. Technically one chooses an Ansatz for the form of the possible interactions, and project the evolution expression onto this Ansatz. In this way, of course, we loose exactness, but at least we can have a fair desription, provided we have choosen the subset appropriately. In the following we discuss, how can we choose the best subset for the RG evolution.

### 7.7 An alternative form of the exact RG evolution equation

If we consider the action for the IR modes $S_{<}$(cf. 209) , or the corresponding Lagrangian $\mathcal{L}_{<}$as function of (infinitely many) coupling constant, then we can draw up the regularization independence of the physics in a different way. If we calculate the effective action from the above Lagrangian, it will formally depend on the regulator; using a single-scale dependent regulator family we write

$$
\begin{equation*}
\mathcal{L}_{<}(k ;\{g\}) \xrightarrow{\text { IR path integral }} \Gamma[k ;\{g\} ; \Phi] . \tag{238}
\end{equation*}
$$

The condition that we keep the physics unchanged while modifying the regulator can be assured if we let the couplings depend on the scale, and also we must allow that the field variable is rescaled (wave function renormalization, cf. 166). This can be expressed as

$$
\begin{equation*}
\frac{d}{d k} \Gamma\left[k ;\left\{g_{k}\right\} ; Z_{k} \Phi\right]=0 . \tag{239}
\end{equation*}
$$

This is a total derivative, which can be expanded as

$$
\begin{equation*}
\left(\frac{\partial}{\partial k}+\frac{d g_{n}}{d k} \frac{\partial}{\partial g_{n}}+\frac{d Z_{k}}{d k} \Phi \frac{\partial}{\partial \Phi}\right) \Gamma\left[k ;\left\{g_{k}\right\} ; Z_{k} \Phi\right]=0 . \tag{240}
\end{equation*}
$$

Usually it is drawn up as a condition for the $n$-point functions

$$
\begin{equation*}
\Gamma[k ;\{g\} ; \Phi]=\sum_{n} \frac{1}{n!} \int\left(\prod_{i=1}^{n} d x_{i}\right) \Gamma^{(n)}\left(k ;\{g\}, x_{1}, \ldots x_{n}\right) \Phi\left(x_{1}\right) \ldots \Phi\left(x_{n}\right), \tag{241}
\end{equation*}
$$

then we have

$$
\begin{equation*}
\left(\frac{\partial}{\partial k}+\frac{d g_{n}}{d k} \frac{\partial}{\partial g_{n}}+n \frac{d Z_{k}}{d k}\right) \Gamma^{(n)}\left(k ;\{g\}, x_{1}, \ldots x_{n}\right)=0 \tag{242}
\end{equation*}
$$

Multiplying with $k$ and using the definition of the beta-functions we have

$$
\begin{equation*}
\left(\partial_{t}+\beta_{n} \frac{\partial}{\partial g_{n}}+n \gamma\right) \Gamma^{(n)}\left(k ;\{g\}, x_{1}, \ldots x_{n}\right)=0 \tag{243}
\end{equation*}
$$

where

$$
\begin{equation*}
\gamma=\frac{d Z_{t}}{d t} . \tag{244}
\end{equation*}
$$

The practical use of this method is that we keep a finite number of couplings (including wave function renormalization), and then we calculate the same number of physical observables. The requirement of finiteness then yields the corresponding $\beta$ (and $\gamma$ ) functions.

This method have advantages and disadvantages as compared the previous, FRG method. The disadvantage is that while in the FRG method we calculate the derivative of the effective action, and then perform a single integral, in the present method we must perform the path integral for the IR modes, which is a much harder task. The advantage, on the other hand is, that the formalism of the FRG corresponds to a genuine 1-loop integral, and we can not improve it by perturbative means (only with the inclusion of new operators),
in performing the path integral we can use perturbation theory, and then the beta-functions can be obtained in better and better accuracy even for a finite operator set. We also must remark that unfortunately perturbation theory is not appropriate for models where the coupling is dimensionful, so the perturbative improvement only works in the "critical dimension", where the expansion parameter is dimensionless: in $\Phi^{4}$ theory it is 4D. Lower dimensional theories must be approached from the critical dimension as an expansion in dimensions (the $\epsilon$-expansion), where a convergence issue has to be faced again.

### 7.8 Renormalization equation in renormalized perturbation theory

The same line of thought can be applied to the renormalized perturbation theory, where we split the original, bare couplings to renormalized ones and counterterms. We fix physics with fixing the bare action, with the different assignment of the parts of the original bare couplings to renormalized value and counterterm does not alter the physics, at least after taking into account infinite number of orders. This means that physics is insensitive to the renormalization scheme, provided we choose the renormalized coupling accordingly.

This idea can be used in various ways, and one can choose always that scheme where the convergence of the perturbation theory is the fastest, and then project back the result to a reference scheme by renormalization group transformation. It can be very well used to perform finite temperature resummations, but this is not the topic of the present investigation.

Instead we restrict ourselves here to the change of scale $\mu$. Using another scale means that we use a different scheme, and it has to be compensated by changing the renormalized couplings. This leads to scale dependent coupligns $g_{\text {ren }}(\mu)$. The condition that the bare action is unchanged, and so the physics is unchanged

$$
\begin{equation*}
\frac{\partial}{\partial \mu} \Gamma\left[\mu,\left\{g_{r e n}(\mu)\right\}, \zeta(\mu) \Phi\right]=0 \tag{245}
\end{equation*}
$$

where we have to allow, similarly to the case when the UV regulator has changed, a (this time finite) modification in the wave function renormalization. This is the same as 239 with $k \rightarrow \mu$ correspondance. Therefore the consequences are the same, like 242 . In case of $4 \mathrm{D} \Phi^{4}$ model, therefore we have an exact equation:

$$
\begin{equation*}
\left(\mu \frac{\partial}{\partial \mu}+\beta \frac{\partial}{\partial \lambda}+m^{2} \gamma_{m} \frac{\partial}{\partial m^{2}}+n \gamma\right) \Gamma^{(n)}\left(\mu ;\left\{\lambda, m^{2}\right\}, x_{1}, \ldots x_{n}\right)=0 \tag{246}
\end{equation*}
$$

where

$$
\begin{equation*}
\beta=\frac{\partial \lambda}{\partial \ln \mu}, \quad \gamma_{m}=\frac{\partial \ln m^{2}}{\partial \ln \mu}, \quad \gamma=\frac{\partial Z}{\partial \ln \mu} \tag{247}
\end{equation*}
$$

We have to emphasize that in these exact formulae only the renormalizable coupligns are present, while in (242) all the couplings play a role. This is possible because only formally are $g$ and $g_{\text {ren }}$ similar, the physical meaning is radically different. While $g$ are couplings of an UV effective theory, $g_{\text {ren }}$ are computed from the IR observables in a $\mu$ dependent way. So the FRG and the perturbative RG equations correspond to expressing the physics through the UV degrees of freedom and through IR quantities, respectively.

If in the evolution of the effective action we keep only the renormalizable terms, and omit the quadratic divergent part of the mass, then the couplings depend on $\mu$ and $\Lambda$ as $\ln \Lambda / \mu$. Using the relation of the bare and renormalized couplings as

$$
\begin{equation*}
g_{b a r e}(\Lambda)=g_{r e n}(\mu)+\delta g\left(\frac{\Lambda}{\mu}\right) \tag{248}
\end{equation*}
$$

and taking derivatives with respect to $\Lambda$ and $\mu$ we find:

$$
\begin{equation*}
\frac{\partial g_{\text {bare }}}{\partial \ln \Lambda}=\frac{\partial \delta g}{\partial \ln \Lambda}=-\frac{\partial \delta g}{\partial \ln \mu}=\frac{\partial g_{\text {ren }}}{\partial \ln \mu} \tag{249}
\end{equation*}
$$

so the $\beta$-functions are the same in the two cases.
As we mentioned before, despite the formal equality of the beta functions, the FRG, keeping only the renormalizable terms is just an approximation, the perturbative RG with the renormalized couplings of the renormalizable operators is "exact", provided we take into account all orders. But the meaning of "exactness" is a sensitive question. The perturbative expansion, as all power series, have its own domain of applicaibility. The naive perturbation theory is even not a convergent series, only "asymptotically convergent" which means
that even after taking into account infinite orders, there can be contributions that are missing from the analysis. After a clever resummation, ie. choosing renormalization scheme in a way that a given observable converges the best (eventually even after a finite order), the perturbative expansion can be made convergent, but then we have to deal with the relation between the different schemes, which can be also nonperturbative. All in all, the consequences drawn from the perturbative RG must be treated with care, and the divergences they provide should not necessarily assign to the theory itself, they may be caused by the bad convergence properties of the perturbation theory itself.

### 7.9 LPA and LPA' approximation

Let us return to the FRG equations. As we mentioned, in practice we write up an analytic Ansatz which is analytically treatable. The most simple Ansatz is when we consider only powers of $\Phi$, but not allow any derivatives to appear (apart from the free $(\partial \Phi)^{2}$ term). This is called the Local Potential Approximation (LPA) approximation. We may also allow in addition a wave function renormalization, then we arrive at the LPA' approximation.

To generalize these approaches we can speak about the Local Vertex Approximation (LVA), where the momentum dependence of the quadratic part can be arbitrary, but higher order vertices are local. For a general theroy with $\Phi_{a}(x)$ local fields the corresponding Ansatz reads

$$
\begin{equation*}
\Gamma_{>}[\Phi]=\frac{1}{2} \int \frac{d^{4} p}{(2 \pi)^{4}} \Phi_{a}(-p) \mathcal{K}_{k, a b}(p) \Phi_{b}(p)+\int d^{4} x U_{k}(\Phi(x)) \tag{250}
\end{equation*}
$$

where $U$ is just a function (not functional) of the fields. To make distinction between local and non-local quadratic part, we require $\mathcal{K}(p=0)=0$. Using the reality of the fields $\left(\Phi^{*}(p)=\Phi(-p)\right.$, and the symmetry of the quadratic form we find

$$
\begin{equation*}
\mathcal{K}_{a b}(p)=\mathcal{K}_{b a}^{*}(p)=\mathcal{K}_{b a}(-p) \tag{251}
\end{equation*}
$$

In the followings we will suppress the notation of the $k$ dependence.
Now we can perform derivatons on the Ansatz 250. We use the symmetry property 251 of the kernel, and the fact that

$$
\begin{equation*}
\frac{\partial \Phi_{b}(p)}{\partial \Phi_{a}(x)}=e^{i p x} \delta_{a b} \tag{252}
\end{equation*}
$$

to find

$$
\begin{align*}
& \Gamma_{>, a}^{(1)}(x)=\int \frac{d^{4} p}{(2 \pi)^{4}} e^{-i p x} \mathcal{K}_{a b}(p) \Phi_{b}(p)+U_{a}^{(1)}(\Phi(x)) \\
& \Gamma_{>, a b}^{(2)}(x, y)=\int \frac{d^{4} p}{(2 \pi)^{4}} e^{-i p(x-y)} \mathcal{K}_{a b}(p)+\delta(x-y) U_{a b}^{(2)}(\Phi(x)) \tag{253}
\end{align*}
$$

The exact FRG 223 equation would read in this Ansatz

$$
\begin{equation*}
\partial_{k} \Gamma_{>}=\frac{1}{2} \hat{\partial}_{k} \sum_{i} \ln \left(\Gamma_{>}^{(2)}+R\right)_{i i} \tag{254}
\end{equation*}
$$

where $i$ represents all (internal and spatial) indices.

### 7.9.1 The evolution of the potential

We evaluate the FRG equation at constant argument $\Phi(x)=\Phi_{0}$, then the derivatives of the potential become constant, and we can take the Fourier form of the derivatives

$$
\begin{equation*}
\Gamma_{>, a b}^{(2)}(p)=\mathcal{K}_{a b}(p)+U_{a b}^{(2)} \tag{255}
\end{equation*}
$$

If we write the trace over the momenta we have in 1D $p_{i}=2 \pi / L$, where $L$ is the linear size. In $d$ dimensions, in (near) infinite system we have

$$
\begin{equation*}
\sum_{i} f\left(p_{i}\right)=V \int \frac{d^{d} p}{(2 \pi)^{d}} f(p) \tag{256}
\end{equation*}
$$

On the other hand the effective action, evaluated at constant field yields $\Gamma\left[\Phi_{0}\right]=V U\left(\Phi_{0}\right)$. Putting it back to the previous expression we find, after simplification with the volume

$$
\begin{equation*}
\partial_{k} U=\frac{1}{2} \hat{\partial}_{k} \int \frac{d^{d} p}{(2 \pi)^{d}} \operatorname{Tr} \ln \left(\mathcal{K}(p)+R(p)+U^{(2)}\right) \tag{257}
\end{equation*}
$$

where the derivatives of the potential are taken with respect to the field components, and the trace is over the internal components. This is a self-contained partial differential equation for the evolution of the effective potential. An alternative form comes from 222

$$
\begin{equation*}
\partial_{k} U=\frac{1}{2} \int \frac{d^{d} p}{(2 \pi)^{d}}\left(\partial_{k} R_{a b}\right) G_{a b}(p), \quad G_{a b}^{-1}(p)=\mathcal{K}_{a b}(p)+R_{a b}(p)+U_{a b}^{(2)} \tag{258}
\end{equation*}
$$

To have a more comfortable form, we can use that usually the kernel is rotational invariant, then we can use $d$ dimensional shperical coordinates (cf. Appendix A.1)

$$
\begin{equation*}
\partial_{k} U=\frac{S_{d}}{2} \hat{\partial}_{k} \int_{0}^{\infty} d p p^{d-1} \operatorname{Tr} \ln \left(\mathcal{K}(p)+R(p)+U^{(2)}\right) \tag{259}
\end{equation*}
$$

### 7.9.2 LPA approximation

In the LPA approximation the kernel is simple: $\mathcal{K}(p)=p^{2}$. To have definit results we will assume some properties. First, we assume that the propagation is diagonal (or made it diagonal after a proper basis choice). Then we have

$$
\begin{equation*}
\partial_{k} U=\frac{S_{d}}{2} \sum_{a} \hat{\partial}_{k} \int_{0}^{\infty} d p p^{d-1} \ln \left(p^{2}+R(p)+U_{a}^{(2)}\right) \tag{260}
\end{equation*}
$$

The regulator, as we have discussed, represents the measurement device, but for practical purposes we may choose a comfortable one. The use of Litim's regulator leads to very simple formulaf ${ }^{3}$. The regultor is

$$
\begin{equation*}
R(p)=\left(k^{2}-p^{2}\right) \Theta\left(k^{2}-p^{2}\right) \tag{261}
\end{equation*}
$$

With this choice we have for the evolution equation of the potential reads

$$
\begin{align*}
\partial_{k} U & =\frac{S_{d}}{2} \sum_{a} \hat{\partial}_{k} \int_{0}^{\infty} d p p^{d-1} \ln \left(p^{2}+\left(k^{2}-p^{2}\right) \Theta\left(k^{2}-p^{2}\right)+U_{a}^{(2)}\right)= \\
& =\frac{S_{d}}{2} \sum_{a} \int_{0}^{\infty} d p p^{d-1} \frac{2 k \Theta\left(k^{2}-p^{2}\right)}{p^{2}+\left(k^{2}-p^{2}\right) \Theta\left(k^{2}-p^{2}\right)+U_{a}^{(2)}}=\frac{S_{d}}{2} \sum_{a} \int_{0}^{k} d p p^{d-1} \frac{2 k}{k^{2}+U_{a}^{(2)}} . \tag{262}
\end{align*}
$$

The integrand is constant, so we have

$$
\begin{equation*}
\partial_{k} U=\Omega_{d} \sum_{a} \frac{k^{d+1}}{k^{2}+U_{a}^{(2)}} \tag{263}
\end{equation*}
$$

We later will use often this form, but it is also instructive to rescale the variables in the potential with the corresponding engineering dimensions. We use the facts that the action is dimensionless, the scale $k$ has energy dimension and $[x]=[k]^{-1}$, therefore

$$
\begin{equation*}
[S]=1=[x]^{d}[U] \quad \Rightarrow \quad[U]=[x]^{-d}=[k]^{d} \tag{264}
\end{equation*}
$$

and

$$
\begin{equation*}
1=\left[\int d^{d} x(\partial \Phi)^{2}\right]=[x]^{d-2}[\Phi]^{2} \Rightarrow \quad[\Phi]=[k]^{d / 2-1} \tag{265}
\end{equation*}
$$

[^2]Therefore we will work with the rescaled variables:

$$
\begin{equation*}
\Phi=k^{d / 2-1} \varphi, \quad U(k, \Phi)=k^{d} u(k, \varphi), \quad t=\ln \frac{k}{k_{0}} \tag{266}
\end{equation*}
$$

with some reference $k_{0}$. This means

$$
\begin{align*}
& \partial_{k} U(k, \Phi)=\left.\partial_{k}\left(k^{d} u\left(k, k^{1-d / 2} \Phi\right)\right)\right|_{\Phi}=d k^{d-1} u+k^{d} \partial_{k} u+k^{d}\left(1-\frac{d}{2}\right) k^{-d / 2} \Phi_{a} \partial_{\varphi_{a}} u= \\
& =k^{d-1}\left(k \partial_{k} u+\left(1-\frac{d}{2}\right) \varphi_{a} \partial_{\varphi_{a}} u+d u\right) \tag{267}
\end{align*}
$$

and

$$
\begin{equation*}
\frac{\partial^{2} U}{\partial \Phi_{a} \partial \Phi_{b}}=k^{2} \frac{\partial^{2} u}{\partial \varphi_{a} \partial \varphi_{b}} \tag{268}
\end{equation*}
$$

Therefore we have

$$
\begin{equation*}
\partial_{t} u=-d u+\sum_{a}\left[\left(\frac{d}{2}-1\right) \varphi_{a} \partial_{\varphi_{a}} u+\frac{\Omega_{d}}{1+u_{a}^{(2)}}\right] \tag{269}
\end{equation*}
$$

We remark that all explicit $k$ dependence disappeared from this equation.

### 7.9.3 LPA' approximation

We will proceed using the LPA' approximation, where the kernel is $\mathcal{K}=Z p^{2}$ with some $k$-dependent coefficient, the wave function renormalization. We will again assume that the propagating modes are diagonal. For the regulator we take Litim's regulator 261 with wave function renormalization extension:

$$
\begin{equation*}
R_{a}(p)=\hat{Z}_{a}\left(k^{2}-p^{2}\right) \Theta\left(k^{2}-p^{2}\right) \tag{270}
\end{equation*}
$$

We denoted the coefficient by $\hat{Z}$ in order to remind us that it is a part of the regulator, but later we will choose $\hat{Z}=Z$. With this choice we have for the evolution equation of the potential

$$
\begin{align*}
\partial_{k} U & =\frac{S_{d}}{2} \sum_{a} \hat{\partial}_{k} \int_{0}^{\infty} d p p^{d-1} \ln \left(Z_{a} p^{2}+\hat{Z}_{a}\left(k^{2}-p^{2}\right) \Theta\left(k^{2}-p^{2}\right)+U_{a}^{(2)}\right)= \\
& =\frac{S_{d}}{2} \sum_{a} \int_{0}^{\infty} d p p^{d-1} \frac{\partial_{k} \hat{Z}_{a}\left(k^{2}-p^{2}\right) \Theta\left(k^{2}-p^{2}\right)+2 k \hat{Z}_{a} \Theta\left(k^{2}-p^{2}\right)}{Z_{a} p^{2}+\hat{Z}_{a}\left(k^{2}-p^{2}\right) \Theta\left(k^{2}-p^{2}\right)+U_{a}^{(2)}}= \\
& =\frac{S_{d}}{2} \sum_{a} \int_{0}^{k} d p p^{d-1} \frac{\partial_{k} \hat{Z}_{a}\left(k^{2}-p^{2}\right)+2 k \hat{Z}_{a}}{Z_{a} k^{2}+U_{a}^{(2)}} . \tag{271}
\end{align*}
$$

We introduce the following quantity

$$
\begin{equation*}
\eta_{a}=-\frac{\partial \ln Z_{a}}{\partial \ln k}, \tag{272}
\end{equation*}
$$

which will be interpreted as anomalous dimension later. Now we use it to express the derivative $\partial_{k} \hat{Z}=$ $-\eta Z / k$. Then we write

$$
\begin{equation*}
\partial_{k} U=\frac{S_{d}}{2} \sum_{a} \frac{2 k Z_{a}}{Z_{a} k^{2}+U_{a}^{(2)}} \int_{0}^{k} d p p^{d-1}\left(1-\frac{\eta_{a}}{2}\left(1-\frac{p^{2}}{k^{2}}\right)\right)=\Omega_{d} \sum_{a} \frac{Z_{a} k^{d+1}}{Z_{a} k^{2}+U_{a}^{(2)}}\left(1-\frac{\eta_{a}}{d+2}\right) . \tag{273}
\end{equation*}
$$

For $Z_{a}=1$ and $\eta_{a}=0$ this is equal to the LPA case 263). To have an even closer similarity we can do rescaling.

In the first case we just rescale with $Z_{a}$.

$$
\begin{equation*}
\tilde{\Phi}_{a}=Z_{a}^{1 / 2} \Phi_{a}, \quad U(k, \Phi)=\tilde{U}(k, \tilde{\Phi}) \tag{274}
\end{equation*}
$$

Then we have

$$
\begin{equation*}
\frac{\partial^{2} U}{\partial \Phi_{a}^{2}}=Z_{a} \frac{\partial^{2} \tilde{U}}{\partial \tilde{\Phi}_{a}^{2}}, \quad \partial_{k} U=\partial_{k} \tilde{U}+\frac{1}{2} \frac{\partial_{k} Z_{a}}{Z_{a}^{1 / 2}} \Phi_{a} \frac{\partial \tilde{U}}{\partial \tilde{\Phi}_{a}}=\partial_{k} \tilde{U}-\frac{\eta_{a}}{2 k} \tilde{\Phi}_{a} \frac{\partial \tilde{U}}{\partial \tilde{\Phi}_{a}}, \tag{275}
\end{equation*}
$$

then we can get rid of all explicit $Z_{a}$ factors:

$$
\begin{equation*}
\partial_{k} \tilde{U}=\frac{\eta_{a}}{2 k} \tilde{\Phi}_{a} \tilde{U}^{(1)}+\Omega_{d} \sum_{a} \frac{k^{d+1}}{k^{2}+\tilde{U}_{a}^{(2)}}\left(1-\frac{\eta_{a}}{d+2}\right) \cdot \tag{276}
\end{equation*}
$$

Another rescaling method is when we rescale the engineering dimensions, similiarly to 266)

$$
\begin{equation*}
\varphi_{a}=Z^{1 / 2} k^{1-d / 2} \Phi_{a}, \quad U(k, \Phi)=k^{d} u(k, \varphi), \quad t=\ln \frac{k}{k_{0}} . \tag{277}
\end{equation*}
$$

In this form we can understand the origin of the name of anomalous dimension. If $\eta=$ constant, then $Z \sim k^{-\eta}$; replacing it back to the previous form

$$
\begin{equation*}
\varphi_{a} \sim k^{1-\frac{d+\eta}{2}} \Phi_{a} \tag{278}
\end{equation*}
$$

which is formally the same as if the spacetime dimension would be $d+\eta$.
With the above form

$$
\begin{equation*}
\frac{\partial^{2} U}{\partial \Phi_{a}^{2}}=Z_{a} k^{2} \frac{\partial^{2} u}{\partial \varphi_{a}^{2}}, \quad k \partial_{k} U=k \partial_{k} u+d u+\frac{1}{2} k \partial_{k}\left(Z^{1 / 2} k^{1-d / 2}\right) \Phi_{a} \frac{\partial u}{\partial \varphi_{a}}=k \partial_{k} u+d u-\left(1-\frac{d+\eta_{a}}{2}\right) \varphi_{a} u_{a}^{(1)} . \tag{279}
\end{equation*}
$$

Replacing it back we obtain

$$
\begin{equation*}
\partial_{t} u=-d u+\sum_{a}\left[\left(1-\frac{d+\eta_{a}}{2}\right) \varphi_{a} u_{a}^{(1)}+\frac{\Omega_{d}}{1+u_{a}^{(2)}}\left(1-\frac{\eta_{a}}{d+2}\right)\right] . \tag{280}
\end{equation*}
$$

### 7.9.4 Evolution of the kernel

To give an account for the evolution of the kernel we have to compute the higher derivatives, too:

$$
\begin{equation*}
\Gamma_{a b c}^{(3)}=U_{a b c}^{(3)}, \quad \Gamma_{a b c d}^{(4)}=U_{a b c d}^{(4)}, \tag{281}
\end{equation*}
$$

which are constant if we evaluate them at constant background. Nevertheless, these couplings can contain derivaties, as it is the case for example in the gauge theories.

Now we can write up (228) at static arguments. It is best to perform a Fourier transformation and then we have

$$
\begin{equation*}
\partial_{k}\left(\mathcal{K}_{a b}(p)+U_{a b}^{(2)}\right)=\frac{1}{2} \hat{\partial}_{k} \int \frac{d^{d} q}{(2 \pi)^{d}}\left[U_{a b c d}^{(4)}(p, q) G_{d c}(q)-U_{a c d}^{(3)}(p, q) U_{b e f}^{(3)}(p, q) G_{d e}(q) G_{f c}(\ell)\right], \tag{282}
\end{equation*}
$$

where $\ell=q-p$. We can now subtract the $p=0$ expression, and use that $\mathcal{K}(p=0)=0$. We have

$$
\begin{equation*}
\partial_{k} \tilde{\mathcal{K}}_{a b}(p)=\frac{1}{2} \hat{\partial}_{k} \int \frac{d^{d} q}{(2 \pi)^{d}}\left[U_{a b c d}^{(4)}(p, q) G_{d c}(q)-U_{a c d}^{(3)}(p, q) U_{b e f}^{(3)}(p, q) G_{d e}(q) G_{f c}(\ell)\right], \tag{283}
\end{equation*}
$$

and $\mathcal{K}_{a b}(p)=\tilde{\mathcal{K}}_{a b}(p)-\tilde{\mathcal{K}}_{a b}(0)$.
In the rest of this subsection we will assume that $U^{(3)}$ and $U^{(4)}$ do not contain derivatives, then the term containg the $U^{(4)}$ term is constant, and so it drops out from the evolution of the kernel. Moreover, we use LPA' approximation, and diagonal representation. Then the kernel is $\mathcal{K}_{a b}(p)=Z_{a} \delta_{a b} p^{2}$, and the propagators are diagonal, too. The running of the wave function renormalization constant can be obtained after twofold differentiation with respect to $p_{\mu}$ and $p_{\nu}$, since $\partial_{\mu} \partial_{\nu} Z p^{2}=2 Z \delta_{\mu \nu}$. So we find

$$
\begin{equation*}
\partial_{k} Z_{a} \delta_{a b} \delta_{\mu \nu}=-\left.\frac{1}{4} U_{a c d}^{(3)} U_{b d c}^{(3)} \hat{\partial}_{k} \int \frac{d^{d} q}{(2 \pi)^{d}} G_{d}(q) \frac{\partial^{2} G_{c}(q-p)}{\partial p_{\mu} \partial p_{\nu}}\right|_{p=0} \tag{284}
\end{equation*}
$$

The left and right hand sides are consistent only, if for a given $d c$ index pair only one field can couple. In this case $U_{a c d}^{(3)} U_{b d c}^{(3)}=\delta_{a b}\left(U_{a c d}^{(3)}\right)^{2}$, and so we have

$$
\begin{equation*}
\partial_{k} Z_{a} \delta_{\mu \nu}=-\left.\frac{1}{4}\left(U_{a c d}^{(3)}\right)^{2} \hat{\partial}_{k} \int \frac{d^{d} q}{(2 \pi)^{d}} G_{d}(q) \frac{\partial^{2} G_{c}(q-p)}{\partial p_{\mu} \partial p_{\nu}}\right|_{p=0} \tag{285}
\end{equation*}
$$

Now we can use

$$
\begin{equation*}
\left.\frac{\partial^{2} G_{f c}(q-p)}{\partial p_{\mu} \partial p_{\nu}}\right|_{p=0}=\frac{\partial^{2} G_{f c}(q)}{\partial q_{\mu} \partial q_{\nu}} \tag{286}
\end{equation*}
$$

and we can perform a partial integration to obtain

$$
\begin{equation*}
\partial_{k} Z_{a} \delta_{\mu \nu}=\frac{1}{4}\left(U_{a c d}^{(3)}\right)^{2} \hat{\partial}_{k} \int \frac{d^{d} q}{(2 \pi)^{d}} \frac{\partial G_{d}(q)}{\partial q_{\mu}} \frac{\partial G_{c}(q)}{\partial q_{\nu}} . \tag{287}
\end{equation*}
$$

Because the propagator depends on $q^{2}$, and $\partial_{\mu} G\left(q^{2}\right)=2 q_{\mu} G^{\prime}\left(q^{2}\right)$, we can write

$$
\begin{equation*}
\partial_{k} Z_{a} \delta_{\mu \nu}=\left(U_{a c d}^{(3)}\right)^{2} \hat{\partial}_{k} \int \frac{d^{d} q}{(2 \pi)^{d}} q_{\mu} q_{\nu} \frac{\partial G_{c}(q)}{\partial q^{2}} \frac{\partial G_{d}(q)}{\partial q^{2}} \tag{288}
\end{equation*}
$$

Taking trace in $\mu \nu$ and dividing the result by $d$ we obtain

$$
\begin{equation*}
\partial_{k} Z_{a}=\frac{1}{d}\left(U_{a c d}^{(3)}\right)^{2} \hat{\partial}_{k} \int \frac{d^{d} q}{(2 \pi)^{d}} q^{2} \frac{\partial G_{c}(q)}{\partial q^{2}} \frac{\partial G_{d}(q)}{\partial q^{2}} \tag{289}
\end{equation*}
$$

Now we can change to spherical coordinates in $d$ dimensions, and use the result of Appendix A. 1 to write

$$
\begin{equation*}
\partial_{k} Z_{a}=\Omega_{d}\left(U_{a c d}^{(3)}\right)^{2} \hat{\partial}_{k} \int_{0}^{\infty} d q q^{d+1} \frac{\partial G_{d}(q)}{\partial q^{2}} \frac{\partial G_{c}(q)}{\partial q^{2}} \tag{290}
\end{equation*}
$$

With Litim's regulator we have

$$
\begin{equation*}
G_{a}(q)=\frac{1}{Z_{a} q^{2}+\hat{Z}_{a}\left(k^{2}-q^{2}\right) \Theta\left(k^{2}-q^{2}\right)+U_{a}^{(2)}} \Rightarrow \frac{d G_{a}}{d q^{2}}=-\frac{Z_{a}-\hat{Z}_{a} \Theta\left(k^{2}-q^{2}\right)}{\left(Z_{a} q^{2}+\hat{Z}_{a}\left(k^{2}-q^{2}\right) \Theta\left(k^{2}-q^{2}\right)+U_{a}^{(2)}\right)^{2}} \tag{291}
\end{equation*}
$$

Then it is

$$
\begin{equation*}
\left.\frac{d G_{a}}{d q^{2}}\right|_{q<k}=-\frac{Z_{a}-\hat{Z}_{a}}{\left(Z_{a} q^{2}+\hat{Z}_{a}\left(k^{2}-q^{2}\right)+U_{a}^{(2)}\right)^{2}},\left.\quad \frac{d G_{a}}{d q^{2}}\right|_{q>k}=-\frac{Z_{a}}{\left(Z_{a} q^{2}+U_{a}^{(2)}\right)^{2}} . \tag{292}
\end{equation*}
$$

So we can separate the integral according to $k$ :

$$
\begin{equation*}
\partial_{k} Z_{a}=\Omega_{d}\left(U_{a c d}^{(3)}\right)^{2} \partial_{k}\left[\int_{0}^{k} d q q^{d+1} \frac{Z_{c}-\hat{Z}_{c}}{\mathcal{K}_{r e g, c}^{2}(k, q)} \frac{Z_{d}-\hat{Z}_{d}}{\mathcal{K}_{r e g, d}^{2}(k, q)}+\int_{k}^{\infty} d q q^{d+1} \frac{Z_{c}}{\left(Z_{c} q^{2}+U_{c}^{(2)}\right)^{2}} \frac{Z_{d}}{\left(Z_{d} q^{2}+U_{d}^{(2)}\right)^{2}}\right] \tag{293}
\end{equation*}
$$

where the denominators of the first two terms are denoted by $\mathcal{K}_{\text {reg }}$. Now the point is that in the first term the derivative may act on any $k$ function, there still remains one $Z-\hat{Z}$ term that renders the complete contribution zero. So only the second term remains, where the only $k$-dependence is at the integration limit:

$$
\begin{equation*}
\partial_{k} Z_{a}=-\Omega_{d}\left(U_{a c d}^{(3)}\right)^{2} \frac{Z_{c} Z_{d} k^{d+1}}{\left(Z_{c} k^{2}+U_{c}^{(2)}\right)^{2}\left(Z_{d} k^{2}+U_{d}^{(2)}\right)^{2}} \tag{294}
\end{equation*}
$$

If we change to $\tilde{U}$, which also means $\left(U_{a c d}^{(3)}\right)^{2}=Z_{a} Z_{b} Z_{c}\left(\tilde{U}_{a c d}^{(3)}\right)^{2}$. So we have

$$
\begin{equation*}
\eta_{a}=\Omega_{d}\left(\tilde{U}_{a c d}^{(3)}\right)^{2} \frac{k^{d+2}}{\left(k^{2}+\tilde{U}_{c}^{(2)}\right)^{2}\left(k^{2}+\tilde{U}_{d}^{(2)}\right)^{2}} \tag{295}
\end{equation*}
$$

With $u$ we find

$$
\begin{equation*}
\frac{\partial^{3} U}{\partial \Phi_{a} \partial \Phi_{b} \partial \Phi_{c}}=k^{3-\frac{d}{2}} Z_{a}^{1 / 2} Z_{b}^{1 / 2} Z_{c}^{1 / 2} \frac{\partial^{3} u}{\partial \varphi_{a} \partial \varphi_{b} \partial \varphi_{c}} \tag{296}
\end{equation*}
$$

which implies

$$
\begin{equation*}
\eta_{a}=\frac{\Omega_{d}\left(u_{a c d}^{(3)}\right)^{2}}{\left(1+u_{c}^{(2)}\right)^{2}\left(1+u_{d}^{(2)}\right)^{2}} \tag{297}
\end{equation*}
$$

In principle this form can be replaced back to 280 thus giving a closed form for the differential equation. We remark that if the regulator is not so simple or the 3-point function contains a derivative, then the expression of $\eta$ is much more complicated.

## 8 FRG in the scalar model

Let us apply what was said above for $\Phi^{4}$ model: in this case there is only one component. We will use the LPA approximation for simplicity, LPA' in this case only slightly modifies the numerical results.

The $k$-running of the effective potential is described by 263 , now it reads

$$
\begin{equation*}
\partial_{k} U=\Omega_{d} \frac{k^{d+1}}{k^{2}+\partial_{\Phi}^{2} U} \tag{298}
\end{equation*}
$$

Actually, with a simple rescaling we can get rid of the $\Omega_{d}$ factor, too:

$$
\begin{equation*}
U(\Phi)=\Omega_{d} V(\varphi), \quad \Phi=\sqrt{\Omega_{d}} \varphi \tag{299}
\end{equation*}
$$

then the second derivative remains the same, so only $\Omega_{d}$ will drop out

$$
\begin{equation*}
\partial_{k} V=\frac{k^{d+1}}{k^{2}+\partial_{\varphi}^{2} V} \tag{300}
\end{equation*}
$$

We can also treat the rescaled FRG equation 269

$$
\begin{equation*}
\partial_{t} \tilde{u}=-d \tilde{u}+\left(\frac{d}{2}-1\right) \varphi \partial_{\varphi} \tilde{u}+\frac{\Omega_{d}}{1+\partial_{\varphi}^{2} \tilde{u}} \tag{301}
\end{equation*}
$$

Here we also can get rid of $\Omega_{d}$ by $u(\varphi)=\Omega_{d} u\left(\Omega_{d}^{-1 / 2} \varphi\right)$, then we obtain

$$
\begin{equation*}
\partial_{t} u=-d u+\left(\frac{d}{2}-1\right) \varphi \partial_{\varphi} u+\frac{1}{1+\partial_{\varphi}^{2} u} \tag{302}
\end{equation*}
$$

The big advantage of the rescaled equaitons is that there is no explicit $t$-dependence on the right hand side.
The initial condition is that at a certain scale we give the potential. For the sake of simplicity we choose

$$
\begin{equation*}
V(k=\Lambda, \varphi)=\frac{m_{0}^{2}}{2} \varphi^{2}+\frac{\lambda_{0}}{24} \varphi^{4} \tag{303}
\end{equation*}
$$

ie. the classical $\Phi^{4}$ theory. We will examine later what is the effect if we add new terms into the initial condition.

## 8.1 $\Phi^{4}$ model, non rescaled form

As a first approximation we only keep those terms that are present in the original potential, ie. we work with the $\Phi^{4}$ theory. To be able to treat nonzero vacuum expectation values we expand the potential around a nontrivial, $k$-dependent $\Phi_{k}$ field:

$$
\begin{equation*}
V(\varphi)=V_{0 k}+\frac{m_{k}^{2}}{2}\left(\varphi^{2}-\Phi_{k}^{2}\right)+\frac{\lambda_{k}}{24}\left(\varphi^{2}-\Phi_{k}^{2}\right)^{2} \tag{304}
\end{equation*}
$$

We here assumed that the $\varphi \rightarrow-\varphi$ symmetry remains true, and so only even powers of $\varphi$ may appear. The expansion parameter therefore will be

$$
\begin{equation*}
\varrho=\frac{1}{2}\left(\varphi^{2}-\Phi^{2}\right), \tag{305}
\end{equation*}
$$

where the $1 / 2$ is the usual convention.
The $k$-derivative is, expressing it with $\varrho$

$$
\begin{equation*}
\partial_{k} V=\partial_{k} V_{0}-\frac{1}{2} m_{k}^{2} \partial_{k} \Phi_{k}^{2}+\partial_{k} m_{k}^{2} \varrho-\frac{\lambda_{k}}{6} \partial_{k} \Phi_{k}^{2} \varrho+\frac{1}{6} \varrho^{2} \partial_{k} \lambda \tag{306}
\end{equation*}
$$

The second derivative reads as a function of $\varrho$

$$
\begin{equation*}
\partial_{\varphi}^{2} V=m_{k}^{2}+\frac{\lambda_{k}}{3} \Phi_{k}^{2}+\lambda_{k} \varrho \tag{307}
\end{equation*}
$$

The right hand side of the FRG equation (300) reads

$$
\begin{equation*}
\frac{k^{d+1}}{\omega_{k}^{2}+\lambda_{k} \varrho}=\frac{k^{d+1}}{\omega_{k}^{2}}-\frac{k^{d+1} \lambda_{k}}{\omega_{k}^{4}} \varrho+\frac{k^{d+1} \lambda_{k}^{2}}{\omega_{k}^{6}} \varrho^{2}+\ldots \tag{308}
\end{equation*}
$$

By matching the corresponding $\varrho$ powers yields:

$$
\begin{equation*}
\partial_{k} m_{k}^{2}-\frac{\lambda_{k}}{6} \partial_{k} \Phi_{k}^{2}=-\frac{k^{d+1} \lambda_{k}}{\omega_{k}^{4}}, \quad \partial_{k} \lambda_{k}=\frac{6 k^{d+1} \lambda_{k}^{2}}{\omega_{k}^{6}} \quad\left(\omega_{k}^{2}=k^{2}+m_{k}^{2}+\frac{1}{3} \lambda_{k} \Phi_{k}^{2}\right) \tag{309}
\end{equation*}
$$

We do not write up equation for $V_{0}$, since it is not coupled to the other variables.
These are too few equations, since we have 4 variables. Indeed, the position of the minimum was not fixed. We can determine an additional equation by requiring that $\Phi_{k}$ is the minimum, with other words the minimum is at $\varrho=0$ :

$$
\begin{equation*}
0=\partial_{\varphi} V=\varphi\left[m_{k}^{2}+\frac{\lambda_{k}}{3} \varrho\right]=\varphi m_{k}^{2} \tag{310}
\end{equation*}
$$

This means that the condition for the minimum is either $\varphi=0$ or $m_{k}^{2}=0$. The first one is the symmetric case, the second one is the spontaneous symmetry broken (SSB) case. We will discuss it in the next two subsections. We can also choose any other condition for the relation of $\Phi_{k}$ and $m_{k}^{2}$ : we will se such an example, too.

Before we solve them we may determine some characteristic features of the equation

- the coupling constant equation can be written as

$$
\begin{equation*}
\partial_{k} \frac{1}{\lambda_{k}}=-\frac{6 k^{d+1}}{\omega_{k}^{6}}<0 \tag{311}
\end{equation*}
$$

which means that $1 / \lambda_{k}$ decreases when $k$ increases. Since we are starting the evolution at maximal $k$, we find that with decreasing $k$ the $1 / \lambda$ increases, so the coupling itself decreases;

- assuming that at $k \approx 0$ the couplings reach some constant values, then the form of the equations is

$$
\begin{equation*}
\partial_{k} Y \sim \frac{k^{d+1}}{M^{2 n}} \tag{312}
\end{equation*}
$$

where $M^{2}=m_{0}^{2}+\lambda_{0} / 3 \Phi_{0}^{2}$, and $Y$ is any quantity. This means

$$
\begin{equation*}
Y(k \rightarrow 0) \sim Y(0)+c k^{d+2} \tag{313}
\end{equation*}
$$

where $c$ is some constant. This means that at $k \rightarrow 0$ the running becomes very slow, at $d=4$ the scale dependence is only $\sim k^{6}$. The scale where we expect to observe stronger running is when $k^{2} \sim m_{k}^{2}$, ie. when the scale dependence of the numerator starts to dominate;

- we do not expect any singular behaviour starting from the UV and evolving towards the IR;
- but if we want to extend the domain to $k>k_{0}$ values we may get into trouble. Assuming that here $m_{k} \ll k$ in the leading order, we obtain the differential equations

$$
\begin{align*}
m_{k} \ll k: & \partial_{k} \frac{1}{\lambda_{k}}=-6 k^{d-5} \quad \Rightarrow \quad \lambda_{k}=\frac{\lambda_{0}}{1-\frac{6 \lambda_{0}}{d-4}\left(k^{d-4}-k_{0}^{d-4}\right)} \\
d=4: & \partial_{k} \frac{1}{\lambda_{k}}=-6 k^{-1} \quad \Rightarrow \quad \lambda_{k}=\frac{\lambda_{0}}{1-6 \lambda_{0} \ln \frac{k}{k_{0}}} . \tag{314}
\end{align*}
$$

So in all $d \leq 4$ dimensions the coupling constant diverges at some scal ${ }^{4}$ at $d=4$ this scale is

$$
\begin{equation*}
k_{L}=k_{0} e^{\frac{1}{6 \lambda_{0}}} \tag{315}
\end{equation*}
$$

The pole behaviour near $k_{L}$ is called Landau-pole. This means that the $\Phi^{4}$ model cannot be extended to arbitrarily high scales, unless $\lambda=0$. This phenomenon is called triviality, and we will return to it later.

### 8.1.1 Symmetric phase

We use here the Ansatz (304) with $\Phi_{k}^{2}=0$, so our evolution equations read, according to (309)

$$
\begin{equation*}
\partial_{k} m_{k}^{2}=-\frac{k^{d+1} \lambda_{k}}{\omega_{k}^{4}}, \quad \partial_{k} \lambda_{k}=\frac{6 k^{d+1} \lambda_{k}^{2}}{\omega_{k}^{6}}, \quad \omega_{k}^{2}=k^{2}+m_{k}^{2} \tag{316}
\end{equation*}
$$

The analytic expectations can be found in the numerical solution, too. The numerical solution is done by Mathematica we may solve it with NDSolve. Using the initial conditions

$$
\begin{equation*}
k_{0}=1, \quad V_{0}=0, \quad m_{0}^{2}=0, \quad \lambda_{0}=1 \tag{317}
\end{equation*}
$$

we obtain the solutions seen on Fig. 5 at $d=4$, but at other dimensions the solution is similar. We see the


Figure 5: The running of the couplings of the $\Phi^{4}$ model in the symmetric phase at $d=4$.
slowing down of the running towards the IR, practically where $k^{2} \sim m_{k}^{2}$, we see that $\lambda_{k}$ grows, $m_{k}^{2}$ decreases when we increase the scale.

### 8.1.2 Broken symmetry case

We use here the Ansatz (304) with $m_{k}^{2}=0$, so our evolution equations read, according to 309)

$$
\begin{equation*}
\partial_{k} \Phi_{k}^{2}=\frac{6 k^{d+1}}{\omega_{k}^{4}}, \quad \partial_{k} \lambda_{k}=\frac{6 k^{d+1} \lambda_{k}^{2}}{\omega_{k}^{6}}, \quad \omega_{k}^{2}=k^{2}+\frac{1}{3} \lambda_{k} \Phi_{k}^{2} \tag{318}
\end{equation*}
$$

[^3]As we see, the derivative of $\Phi_{k}^{2}$ is positive, so it is decreasing towards the IR. We can choose such initial conditions, when it remains in the broken phase, like

$$
\begin{equation*}
k_{0}=1, \quad \Phi_{1}=1.7, \quad \lambda_{1}=1 \tag{319}
\end{equation*}
$$

then we obtain a picture shown in Fig. 6


Figure 6: The running of the couplings of the $\Phi^{4}$ model in the broken phase at $d=4$.
We can also start from an initial condition, where we end the running in the symmetric phase, cf. Fig. 7. Here we should shift from one regime to the other.


Figure 7: The running of the couplings of the $\Phi^{4}$ model at $d=4$. Here $M^{2}=m^{2}+\lambda \Phi^{2} / 3$.
We can see that there is a scale, where the mass disappears, in this scale the correlation length is infinity, we are in the phase transition regime.

In these plot, like in all plots before, there is a scale in the IR, and this stops the running. In ceratain sense this is the basic "working mode" of the FRG: the coefficient of the potential are modified somewhat, but nothing too interesting happens. For example in the QED the coupling constant $\alpha=e^{2} /\left(4 \pi \varepsilon_{0} \hbar c\right)$ (it is dimensionless) reaches $1 / 137$ value in the IR, at the scale $k=80 \mathrm{GeV}$ (it is the Z-boson mass) it is about $1 / 127$, so it decreased by $\sim 8 \%$.. In principle it will reach the Landau-pole, but it is at very large energies $k=10^{279} \mathrm{GeV}$.

### 8.1.3 The physical values

In the numerical calculation all coefficients have a dimensionless number as their value. How do then we know, how large is the cutoff in real energy units (for example in J or in eV )? As we have seen before, we started the running from $k_{0}=1$. This means that we have chosen the value of $k_{0}$ as the energy unit, i.e. each energy-like quantity is measured in the units of $k_{0}$. Therefore if we get a value for the mass at $k=0$,
it in fact gives the ratio $M^{2} / k_{0}^{2}$. In a real model we measure the value of a particle in physical units, for example in GeV . This sets the physical value for $M^{2}$, then also the UV scale can be computed in GeV.

Consider, for example, the case of Fig. 7. The value of the particle mass $m^{2}=0.03$ in computer units. Thus we can say that $m^{2} / k_{0}^{2}=0.03$, therefore $k_{0}=5.75 m$. If $m$ happens to be the Higgs mass, then in physical units it is $m=m_{H}=125 \mathrm{GeV}$. The value of the cutoff in physical units is therefore $k_{0} \approx 720 \mathrm{GeV}$.

### 8.2 The rescaled equation

We can also determine the equations in the rescaled version (302). Let us choose the Ansatz

$$
\begin{equation*}
u=u_{0}+\frac{\mu^{2}}{2}\left(\varphi^{2}-v^{2}\right)+\frac{\lambda}{24}\left(\varphi^{2}-v^{2}\right)^{2} \tag{320}
\end{equation*}
$$

With the same procedure as before we arrive at the RG equations

$$
\begin{equation*}
\partial_{t} \mu^{2}-\frac{\lambda}{6} \partial_{t} v^{2}=-2 \mu^{2}+\frac{d-2}{6} \lambda v^{2}-\frac{\lambda}{\left(1+M^{2}\right)^{2}}, \quad \partial_{t} \lambda=(d-4) \lambda+\frac{6 \lambda^{2}}{\left(1+M^{2}\right)^{3}}, \quad M^{2}=\mu^{2}+\frac{\lambda}{3} v^{2} \tag{321}
\end{equation*}
$$

In fact numerically we can treat both equations together with a supplementary one

$$
\begin{equation*}
\left(1-\tanh \left(C v^{2}\right)\right) \partial_{t} v^{2}+\left(1+\tanh \left(C v^{2}\right)\right) \partial_{t} \mu^{2}=0 \tag{322}
\end{equation*}
$$

this provides constant $v^{2}$ in the symmetric and constant $\mu^{2}$ in the broken phase.
Now the initial conditions are at $t=0$, and the decreasing $k$ corresponds to negative $t$ values. If we start for example with

$$
\begin{equation*}
v^{2}(0)=1, \quad \mu^{2}(0)=0, \quad \lambda(0)=1 \tag{323}
\end{equation*}
$$

then we obtain the picture shown in Fig. 8 . We remark that in the figure the physical values are plotted, ie.


Figure 8: The running of the couplings of the $\Phi^{4}$ model at $d=4$ from the rescaled equations. Here $M^{2}=\mu^{2}+\lambda v^{2} / 3$.
$\mu^{2} k^{2}=\mu^{2} e^{2 t}$.
The rescaled equations do not contain explicitly the $t$ variable, only through derivatives. This means that in fact the solutions are integral curves of a vector space:

$$
\begin{equation*}
\partial_{t} g_{i}=\beta_{i}(\boldsymbol{g}) \tag{324}
\end{equation*}
$$

If two curves crosses each other, there go on in the same way. This means that different curves can not cross each other.

This vector space can be plotted, this is called flow diagram. In $P h i^{4}$ model it is worth to write up the flow chart for the actual mass term in the SSB phase:

$$
\begin{equation*}
M_{t}^{2}=\frac{\lambda_{t}}{3} v_{t}^{2}, \quad \partial_{t} M_{t}^{2}=-2 M_{t}^{2}+\frac{2 \lambda_{t}\left(1+4 M_{t}^{2}\right)}{\left(1+M_{t}^{2}\right)^{3}} \tag{325}
\end{equation*}
$$

Usualy the flow diagram is directed into the IR, so we should plot the vector field

$$
\begin{equation*}
\boldsymbol{\beta}_{s y m}\left(\mu^{2}, \lambda\right)=\binom{2 \mu^{2}+\frac{\lambda}{\left(1+\mu^{2}\right)^{2}}}{-(d-4) \lambda-\frac{6 \lambda^{2}}{\left(1+M^{3}\right)^{2}}} \quad \boldsymbol{\beta}_{S S B}\left(M^{2}, \lambda\right)=\binom{2 M^{2}-\frac{2 \lambda\left(1+4 M^{2}\right)}{\left(1+M^{2}\right)^{3}}}{-(d-4) \lambda-\frac{6 \lambda^{2}}{\left(1+M^{3}\right)^{2}}} \tag{326}
\end{equation*}
$$

In 4 D we see the corresponding flow diagrams in Fig. 9 . If we analyze the plot in the broken phase (right


Figure 9: The flow diagram of the $4 \mathrm{D} \Phi^{4}$ theory in the symmetric (left) and broken phase (right). The Gaussian fixed point is indicated in red.
panel), then we see that the coupling constant is decreasing, but the mass is either grows, or decreases, depending where we start. If the mass grows then it goes to infinity, corresponding to a constant physical $\operatorname{mass}\left(M_{\text {phys }}^{2}=k^{2} M^{2}\right.$, since we have rescaled with the engineering dimension). If the mass shrinks, then sooner or later it crosses zero, the system gets in the symmetric phase.

There is a point in the flow diagram where the vector fields are zero, this is the $M^{2}=\lambda=0$ point. This is a fixed point of the system, since starting the system at this point it remains there. From the $\lambda$ direction it is attractive, from the $M^{2}$ or $\mu^{2}$ direction it is repelling.

The plot becomes more interesting if we are in $d=3$ in the SSB phase, cf. Fig. 10. There is another


Figure 10: The flow diagram of the $3 \mathrm{D} \Phi^{4}$ theory in the broken phase. Besides the Gaussian fixed point we find another one.
fixed point in the system, and this fixed point plays an important role in the IR physics.

### 8.3 Fixed point

Let us examine the situation when the $\beta$-functions are all zero, these are called fixed points. In the $\Phi^{4}$ model fixed points can be obtained in the rescaled equations.

In the symmetric phase we have $v^{2}=0$, and so

$$
\begin{equation*}
\partial_{t} \mu^{2}=-2 \mu^{2}-\frac{\lambda}{\left(1+m^{2}\right)^{2}}, \quad \partial_{t} \lambda=(d-4) \lambda+\frac{6 \lambda^{2}}{\left(1+m^{3}\right)^{2}} \tag{327}
\end{equation*}
$$

We can see that for $\lambda>0$ the running of the mass consists of two terms with the same sign. So in the symmetric phase there is no fixed point.

In the broken phase $\mu=0$ and

$$
\begin{equation*}
\partial_{t} v^{2}=-(d-2) v^{2}+\frac{6}{\left(1+M^{2}\right)^{2}}, \quad \partial_{t} \lambda=(d-4) \lambda+\frac{6 \lambda^{2}}{\left(1+M^{3}\right)^{2}}, \quad M^{2}=\frac{\lambda}{3} v^{2} \tag{328}
\end{equation*}
$$

The $\beta$-functions can be zero for $2<d<4$ :

$$
\begin{equation*}
v^{2}=\frac{6}{d-2} \frac{1}{\left(1+M^{2}\right)^{2}}, \quad \lambda=\frac{4-d}{6}\left(1+M^{2}\right)^{3} \tag{329}
\end{equation*}
$$

this means

$$
\begin{equation*}
M^{2}=\frac{\lambda}{3} v^{2}=\frac{4-d}{3(d-2)}\left(1+M^{2}\right) \quad \Rightarrow \quad M^{2}=\frac{4-d}{4 d-10} \tag{330}
\end{equation*}
$$

For example for $d=3$ we find

$$
\begin{equation*}
d=3, \quad M^{2}=\frac{1}{2}, \quad \lambda=\frac{9}{16}, \quad v^{2}=\frac{8}{3} \tag{331}
\end{equation*}
$$

This is known as the Wilson-Fischer fixed point.
In the broken phase there is also a 3-point function, as it can be seen by direct differentiation of $u$ by $\varphi$ :

$$
\begin{equation*}
u^{(3)}=\lambda v \rightarrow 0.92 \tag{332}
\end{equation*}
$$

at the fixed point. With the Litims regulator we can also compute the anomalous dimension (297), with the result $\eta=1 /\left(16 \pi^{2}\right)=0.0028$, which is, however, much too small (the experimental value is ${ }^{5} \eta=0.036$ ).

In the fixed point the physical value of the mass $m^{2}=k^{2} \mu^{2}$ is proportional to the scale itself. Therefore the propagator at momentum $k$ is

$$
\begin{equation*}
G(k) \sim \frac{1}{Z k^{2}} \sim \frac{1}{k^{2-\eta}} \tag{333}
\end{equation*}
$$

In general, since the mass is proportional to the scale, all correlation functions are just power law functions.

### 8.4 Effect of higher powers in the potential

In the previous subsection we assumed that the scalar model is written up using terms that is power law and the power is up to $\Phi^{4}$. But nothing forbids to assume also higher order terms. Here we examine what happens when we add a $\Phi^{6}$ term, too.

So our Ansatz will be

$$
\begin{equation*}
V(\varphi)=V_{0 k}+\frac{m_{k}^{2}}{2} \varphi^{2}+\frac{\lambda_{k}}{24} \varphi^{4}+\frac{g_{k}}{720} \varphi^{6} \tag{334}
\end{equation*}
$$

The second derivative reads

$$
\begin{equation*}
\partial_{\varphi}^{2} V=m_{k}^{2}+\frac{\lambda_{k}}{2} \varphi^{2}+\frac{g_{k}}{24} \varphi^{6} \tag{335}
\end{equation*}
$$

We can perform the same procedure as before, but now keeping terms up to $\varphi^{6}$ and find

$$
\begin{align*}
& \partial_{k} V_{0 k}=\frac{k^{d+1}}{\omega_{k}^{2}} \\
& \partial_{k} m_{k}^{2}=-\frac{k^{d+1} \lambda_{k}}{\omega_{k}^{4}} \\
& \partial_{k} \lambda_{k}=\frac{k^{d+1}\left(6 \lambda_{k}^{2}-g_{k} m_{k}^{2}-k^{2} g_{k}\right)}{\omega_{k}^{6}} \\
& \partial_{k} g_{k}=\frac{30 k^{1+d} g_{k} \lambda_{k}}{\omega_{k}^{6}}-\frac{90 k^{1+d} \lambda_{k}^{3}}{\omega_{k}^{8}} \tag{336}
\end{align*}
$$

[^4]
## 9 General properties of the exact RG equation

In this section we overview some general properties of the solutions of the exact RG equation.

### 9.1 Fixed points, relevant and irrelevant operators

In the solution of the FRG equations 223 or 237) a special role is played by that points where the right hand side is zero for all $k$. At these points $\partial_{k} \Gamma=0$, there is no scale dependence whatsoever: we are in a fixed point.

The physical manifestation of a fixed point semms to be very unusual, weird. It means that if we have a microscope and study the structure of the world at small distances we see the same picture. Therefore nothing can have a definite size, all structures have the same properties at all scales. This feature is characteristic to fractals. Surprisingly, however, fractals appear in our world in a lot of cases (like see shores, trees, mathematical fractal contructions, configurations in a second order phase transition), albeit only for a limited range of scale variation. The best known and most studied example of a scale-free system is the world of second order phase transitions.

So let us assume that we study the equation 237) and found a point $g_{i}^{*}$ where $\beta_{n}\left(g^{*}\right)=0 \forall n$. Assuming that $\beta_{n}$ are holomorphic we can expand them around $g^{*}$. Then we find

$$
\begin{equation*}
\partial_{t}\left(g_{n}^{*}+\zeta_{n}\right)=\partial_{t} \zeta_{n}=\beta_{n}\left(g^{*}+\zeta\right)=\left.\frac{\partial \beta_{n}}{\partial g_{m}}\right|_{g^{*}} \zeta_{m}+\mathcal{O}\left(\zeta^{2}\right) \tag{337}
\end{equation*}
$$

We obtain a linear equation in the first approximation

$$
\begin{equation*}
\partial_{t} \zeta_{n}=M_{n m} \zeta_{m}, \quad M_{n m}=\left.\frac{\partial \mathcal{F}_{n}}{\partial g_{m}}\right|_{g^{*}} \tag{338}
\end{equation*}
$$

It is easy to solve this equation, provided we can diagonalize the $M$ matrix. Let us assume that its right eigenvectors are $v^{(a)}$, they form a basis, and the eigenvalues are $\lambda^{(a)}$. We remark that $M$ is dimensionless, so its eigenvalues are just pure numbers. Then we have

$$
\begin{equation*}
M v^{(a)}=\lambda^{(a)} v^{(a)} \Rightarrow \zeta=\sum_{a} z_{a} v^{(a)} \Rightarrow \partial_{t} z_{a}=\lambda^{(a)} z_{a} \tag{339}
\end{equation*}
$$

and this implies

$$
\begin{equation*}
z_{a}(t)=z_{a} e^{\lambda^{(a)}} t, \quad z_{a}(k)=z_{a} k^{\lambda^{(a)}} \tag{340}
\end{equation*}
$$

The effective Lagrangian therefore reads

$$
\begin{equation*}
\mathcal{L}(k ; \Phi)=\mathcal{L}^{*}(k ; \Phi)+\sum_{n} \zeta_{n}(k) O_{n}=\mathcal{L}^{*}(k ; \Phi)+\sum_{a} z_{a} k^{\lambda^{(a)}} \mathcal{A}_{a} \tag{341}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{A}_{a}=\sum_{n} v_{n}^{(a)} O_{n} \tag{342}
\end{equation*}
$$

These $\mathcal{A}_{a}$ operators - themselves forming a new basis - are called scaling operators, and they are characteristic to the given fixed point.

This means that all coefficients depend on the scale as a power law. There can be three different cases:

- $\lambda^{(a)}>0$ : in this case the coefficient will vanish in the IR limit $k \rightarrow 0$. These coupling constants are called irrelevant couplings, the corresponding $\mathcal{A}_{a}$ scaling operators are the irrelevant operators. If we want to study the system near the fixed point, but at large distances, we can simply omit these operators (assuming that they are not play important role in the formation of the fixed point).
- $\lambda^{(a)}<0$ : in this case the coefficient will blow up in the IR limit $k \rightarrow 0$. These coupling constants are called relevant couplings, the corresponding $\mathcal{A}_{a}$ scaling operators are the relevant operators. We must realize that we solve an exact equation, so the growing of these couplings is physical, we can realize it in a physical system as for example the growing correlation length.
- $\lambda^{(a)}=0$ : these operators do not depend on the scale in the linear approximation, so we can not decide whether they disappear or not in the IR limit $k \rightarrow 0$. These coupling constants are called marginal couplings, the corresponding $\mathcal{A}_{a}$ scaling operators are the marginal operators. Usually one needs a more than linear expansion around the fixed point in order to decide whether a marginal operator is relevant or irrelevant.

According to this picture, for a satisfactory description it is enough to follow only the running of the relevant and possibly the marginal operators. In a given theory there is only a finite number of these operators as we will see soon. This is therefore that subset that presumably gives an appropriate representation of the physics.

Nevertheless, the infinite set of irrelevant operators is there, and there are situations, when these have to be taken into account. One such case is when we happen to switch to a different fixed point: there the scaling operators are different, and the new scaling operators will contain contributions from the irrelevant operators of the old fixed point. Another example is when amongst the fixed point couplings there are irrelevant ones: this can happen, since the fixed point and the linear expansion around the fixed point are not depend on each other. Usually this is not the case, or it can be neglected, but there are situations where they play important role: for example when the theory with only including relevant (and marginal) operators has no stable ground state. A third example when we have infinitely many operators, and, although they are irrelevant one-by-one, their cumulative effect is not negligible.

In these cases we have to pay more attention to the irrelevant operators - that are called dangereously irrelevant to warn the danger. In fact it is very hard to identify and even harder to treat these situations.

### 9.2 Gaussian fixed points and expansion around it

There is always a fixed point in field theories, when all the interactions have zero coefficients. Then the theory is free, there is no radiative corrections, the effective action is identical with the classical action. This is called gaussian fixed point, since the $e^{-S_{E}}$ statistical weights are gaussian.

Perturbation theory is nothing else than an expansion around the gaussian fixed point, since we assume that the values of the coupligns are small. Therefore the RG equations can be determined perturbatively from the requirement that physics is unchanged, but the regulator is modified.

We will see explicit examples, but first let us discuss some pecularities of the gaussian fixed point. If there are no radiative corrections, the effective action is equal to the classical action. This means that the regulator dependence shows up first in one-loop order. The change of the coupling, therefore contains higher powers of the couplings, in the linear order there is no regulator dependence. Thus the linearization around the gaussian fixed point does not work, the running of the couplings always depend nonlinearly on the couplings.

To have a feeling we assume that the beta-function for the running of the coupling depends only on the coupling, and we power expand it near zero

$$
\begin{equation*}
\beta(\lambda)=\beta_{1} \lambda^{2}+\mathcal{O}\left(\lambda^{3}\right) \tag{343}
\end{equation*}
$$

since the linear term is missing. Here $\beta_{1}$ is some number, characteristic to the given theory. At lowest order this yields an RG equation, which true in the $\Phi^{4}$ theory as well as in gauge theories for the squared coupling:

$$
\begin{equation*}
\partial_{t} \lambda=\beta_{1} \lambda^{2} \tag{344}
\end{equation*}
$$

We can solve it as (using $\left.t_{0}=\ln k_{0} / k_{0}=0\right)$

$$
\begin{equation*}
\partial_{t} \frac{1}{\lambda}=-\beta_{1} \quad \Rightarrow \quad \frac{1}{\lambda}-\frac{1}{\lambda_{0}}=-\beta_{1} t \tag{345}
\end{equation*}
$$

so finally

$$
\begin{equation*}
\lambda(t)=\frac{\lambda_{0}}{1-\beta_{1} \lambda_{0} t}, \quad \lambda(k)=\frac{\lambda_{0}}{1-\beta_{1} \lambda_{0} \ln \frac{k}{k_{0}}} . \tag{346}
\end{equation*}
$$

Instread of a power-law running we encounter a logarithmic running. We can rewrite this expression as

$$
\begin{equation*}
\lambda(k)=\frac{1}{\beta_{1}} \frac{1}{\frac{1}{\beta_{1} \lambda_{0}}-\ln \frac{k}{k_{0}}}=\frac{1}{-\beta_{1} \ln \frac{k}{k_{0} e^{\frac{1}{\beta_{1} \lambda_{0}}}}} . \tag{347}
\end{equation*}
$$

Introducing

$$
\begin{equation*}
\Lambda_{L}=k_{0} e^{\frac{1}{\beta_{1} \lambda_{0}}} \tag{348}
\end{equation*}
$$

we can write the running as

$$
\begin{equation*}
\lambda(k)=\frac{1}{-\beta_{1} \ln \frac{k}{\Lambda_{L}}} \tag{349}
\end{equation*}
$$

This is a very generic form of the one-loop running, and it has very robust properties. First of all we see that at $k=\Lambda_{L}$ the coupling constant blows up; for $k \approx \Lambda_{L}$ we find

$$
\begin{equation*}
\lambda\left(k \approx \Lambda_{L}\right)=\frac{1}{-\beta_{1} \ln \left(1-\frac{\Lambda_{L}-k}{\Lambda_{L}}\right)}=\frac{1}{\beta_{1}} \frac{\Lambda_{L}}{\Lambda_{L}-k} \tag{350}
\end{equation*}
$$

so there is a pole in the running at this scale. This is called Landau-pole, and $\Lambda_{L}$ is the Landau-scale.
The Landau-scale also determined the domain of the running coupling. If $\beta_{1}>0$ then positive coupling - which is needed for the stability of the $\Phi^{4}$ theory - can only be ensured for $k<\Lambda_{L}$. This means that $\Phi^{4}$ theory (and also QED) can be correctly defined in the IR regime. Moreover, for $k \rightarrow 0$ the coupling goes to zero

$$
\begin{equation*}
\beta_{1}>0 \Rightarrow \lim _{k \rightarrow 0} \lambda(k)=0 \tag{351}
\end{equation*}
$$

which means that at large distances the theory becomes free: this phenomenon is used to call triviality. With other words the quartic coupling, which is marginal (the linear term in the RG equation written up at the gaussian fixed point is zero), but higher orders make it irrelevant. We can say that the gaussian fixed point is IR stable.

We can recover the similarity of the running coupling and the shadowing in the first section. $\Phi^{4}$ theory as well as QED screen their charges, so that at large distances the charges can not be observed any more.

If, however, $\beta_{1}<0$, then the coupling is positive for $k>\Lambda_{L}$. At $k \rightarrow \infty$ we again find

$$
\begin{equation*}
\beta_{1}<0 \Rightarrow \lim _{k \rightarrow \infty} \lambda(k)=0 \tag{352}
\end{equation*}
$$

which means that this theory is free at small distances, which property is called asymptotic freedom. This means that the gaussian fixed point is reached at small scales, so it is UV stable.

What is said is only the lowest order result. But approaching the Landau pole the coupling starts to grow, meaning that the higher order corrections cannot be neglected. There can be different situations there: either there is a zero of the $\beta$-function at some nontrivial coupligns $\lambda^{*}$, or the $\beta$-function remains positive (or negative in the asymptotically free systems). In the first case there is a non-gaussian fixed point discussed earlier, in the second case the fixed point remains gaussian, only the running is modified somewhat, as compared to the one-loop case. In $\Phi^{4}$ theory numerical studies show that the second case is manifested.

### 9.3 Dimensional transmutation

The fact that the coupling constants depend on the scale has an interesting consequence. Let us consider a Lagrangian that provides the action for some quantum or statistical field theory. Imagine that we have some tools to set the value of the coupling, and so we can tune the system from weakly coupled $\lambda \ll 1$ to strongly coupled regime $\lambda \gg 1$.

Interestingly this naive expectation turns out to be false if there is an IR or UV stable Gaussian fixed point in the system. The reason is that when we give $k_{0}$ and $\lambda_{0}$, then we in fact give the Landau-scale $\Lambda_{L}$, according to 348 . If we choose at the scale $k_{0}$ a different coupling, it just changes the value of $\Lambda_{L}$, but
does not influence the (349) running itself. This means that the physics is not changed, just the scale where we see a given physics.

Therefore the coupling looses that interpretation that it determines whether a theory is strongly or weakly interacting, instead its value just sets the scale where the theroy becomes weakly/strongly coupled. This means that the dimensionless coupling in fact sets a dimensionful scale: this phenomenon os called dimensional transmutation.

If the fixed point is non-Gaussian, then the physics is similar, just the setting of the actual value of the coupling constant determines the scale where we arrive at a given distance to the fixed point.

## A Mathematical details

## A. 1 Surface factors and basic integrals

The integral which helps to evaluate the surface factors reads

$$
\begin{equation*}
\int d^{d} x e^{-x^{2}}=K_{d} \int_{0}^{\infty} d x x^{d-1} e^{-x^{2}}=\frac{1}{2} K_{d} \Gamma\left(\frac{d}{2}\right)=\left(\int_{-\infty}^{\infty} d x e^{-x^{2}}\right)^{d}=\pi^{d / 2} \tag{353}
\end{equation*}
$$

It yields

$$
\begin{equation*}
K_{d}=\frac{2 \pi^{d / 2}}{\Gamma\left(\frac{d}{2}\right)}, \quad S_{d}=\frac{K_{d}}{(2 \pi)^{d}}=\frac{2}{(4 \pi)^{d / 2} \Gamma\left(\frac{d}{2}\right)}, \quad \Omega_{d}=\frac{S_{d}}{d}=\frac{1}{(4 \pi)^{d / 2} \Gamma\left(\frac{d}{2}+1\right)} \tag{354}
\end{equation*}
$$

In particular

$$
\begin{equation*}
K_{1}=2, \quad K_{2}=2 \pi, \quad K_{3}=4 \pi, \quad K_{4}=2 \pi^{2} \tag{355}
\end{equation*}
$$

and

$$
\begin{equation*}
S_{1}=\frac{1}{\pi}, \quad S_{2}=\frac{1}{2 \pi}, \quad S_{3}=\frac{1}{2 \pi^{2}}, \quad S_{4}=\frac{1}{8 \pi^{2}} \tag{356}
\end{equation*}
$$

and

$$
\begin{equation*}
\Omega_{1}=\frac{1}{\pi}, \quad \Omega_{2}=\frac{1}{4 \pi}, \quad \Omega_{3}=\frac{1}{6 \pi^{2}}, \quad \Omega_{4}=\frac{1}{32 \pi^{2}} \tag{357}
\end{equation*}
$$

The first integral is

$$
\begin{equation*}
I_{a b}(x)=\int_{0}^{\infty} d z z^{a-1}(x+z)^{-b}=x^{a-b} \frac{\Gamma(b-a) \Gamma(a)}{\Gamma(b)} \tag{358}
\end{equation*}
$$

where $\Gamma$ is the gamma-function.
With the help of this

$$
\begin{align*}
\mathcal{T}_{\alpha} & =\int \frac{d^{d} p}{(2 \pi)^{d}} \frac{1}{\left(p^{2}+m^{2}\right)^{\alpha}}=S_{d} \int_{0}^{\infty} d p \frac{p^{d-1}}{\left(p^{2}+m^{2}\right)^{\alpha}}=\left\{\begin{array}{c}
z=p^{2} \\
d z=2 p d p
\end{array}\right\}=\frac{S_{d}}{2} \int_{0}^{\infty} d z z^{d / 2-1}\left(z+m^{2}\right)^{-\alpha}= \\
& =\frac{S_{d}}{2} m^{d-2 \alpha} \Gamma\left(\alpha-\frac{d}{2}\right) \Gamma\left(\frac{d}{2}\right)=\frac{m^{d-2 \alpha} \Gamma\left(\alpha-\frac{d}{2}\right)}{(4 \pi)^{d / 2}} \tag{359}
\end{align*}
$$

In this notation $\mathcal{T}=\mathcal{T}_{1}$ and $\mathcal{B}_{2}=\mathcal{T}_{2}$.
The 4D tadpole and bubble diagrams with momentum cutoff

$$
\begin{align*}
& \mathcal{T}=\int \frac{d^{4} p}{(2 \pi)^{4}} \frac{1}{p^{2}+m^{2}}=S_{4} \int_{0}^{\Lambda} d p \frac{p^{3}}{p^{2}+m^{2}}=\frac{S_{4}}{2}\left[\Lambda^{2}-m^{2} \ln \frac{\Lambda^{2}+m^{2}}{m^{2}}\right] \\
& \mathcal{B}_{2}=\int \frac{d^{4} p}{(2 \pi)^{4}} \frac{1}{\left(p^{2}+m^{2}\right)^{2}}=S_{4} \int_{0}^{\Lambda} d p \frac{p^{3}}{\left(p^{2}+m^{2}\right)^{2}}=\frac{S_{4}}{2}\left[-\frac{\Lambda^{2}}{m^{2}+\Lambda^{2}}+\ln \frac{\Lambda^{2}+m^{2}}{m^{2}}\right] \tag{360}
\end{align*}
$$

If $\Lambda \gg m$ we find

$$
\begin{align*}
\mathcal{T} & =\int \frac{d^{4} p}{(2 \pi)^{4}} \frac{1}{p^{2}+m^{2}}=\frac{S_{4}}{2}\left[\Lambda^{2}+m^{2} \ln \frac{m^{2}}{\Lambda^{2}}\right] \\
\mathcal{B}_{2} & =\int \frac{d^{4} p}{(2 \pi)^{4}} \frac{1}{\left(p^{2}+m^{2}\right)^{2}}=-\frac{S_{4}}{2}\left[\ln \frac{m^{2}}{\Lambda^{2}}+1\right] \tag{361}
\end{align*}
$$

The momentum dependent bubble in the large $\Lambda$ limit

$$
\begin{align*}
\mathcal{B}_{2}(q) & =\int \frac{d^{4} p}{(2 \pi)^{4}} \frac{1}{\left(p^{2}+m^{2}\right)\left((p-q)^{2}+m^{2}\right)}=\int \frac{d^{4} p}{(2 \pi)^{4}} \int_{0}^{1} d x \frac{1}{\left((1-x)\left(p^{2}+m^{2}\right)+x\left((p-q)^{2}+m^{2}\right)\right)^{2}}= \\
& =\int \frac{d^{4} p}{(2 \pi)^{4}} \int_{0}^{1} d x \frac{1}{\left(p^{2}+m^{2}-2 p q x+x q^{2}\right)^{2}}=\int \frac{d^{4} p}{(2 \pi)^{4}} \int_{0}^{1} d x \frac{1}{\left((p-q x)^{2}+m^{2}+x(1-x) q^{2}\right)^{2}}= \\
& =\int_{0}^{1} d x \int \frac{d^{4} p}{(2 \pi)^{4}} \frac{1}{\left(p^{2}+m^{2}+x(1-x) q^{2}\right)^{2}}=-\frac{S_{4}}{2} \int_{0}^{1} d x\left[\ln \frac{m^{2}+x(1-x) q^{2}}{\Lambda^{2}}+1\right]= \\
& =-\frac{S_{4}}{2}\left[\ln \frac{q^{2}}{\Lambda^{2}}+1+\int_{0}^{1} d x \ln \left(\frac{m^{2}}{q^{2}}+x(1-x)\right)\right]=-\frac{S_{4}}{2}\left[\ln \frac{m^{2}}{\Lambda^{2}}-1+F \ln \frac{F+1}{F-1}\right] \tag{362}
\end{align*}
$$

where

$$
\begin{equation*}
F=\sqrt{1+\frac{4 m^{2}}{q^{2}}} \tag{363}
\end{equation*}
$$

If $q \rightarrow 0$, the last term goes to 2 , and we recover the previous result. If $q \gg m$ then $F=1+2 m^{2} / q^{2}$ and

$$
\begin{equation*}
\mathcal{B}_{2}(q) \xrightarrow{q \gg m}-\frac{S_{4}}{2}\left[\ln \frac{q^{2}}{\Lambda^{2}}-1\right] \tag{364}
\end{equation*}
$$

## A. 2 Integrals

We start with

$$
\begin{equation*}
\int_{-\infty}^{\infty} \frac{d x}{(x-a)(x-b)}=\frac{1}{a-b} \int_{-\infty}^{\infty}\left[\frac{1}{x-a}-\frac{1}{x-b}\right] d x \tag{365}
\end{equation*}
$$

We close the contour from above or from below, the result is the same:

$$
\begin{equation*}
\int_{-\infty}^{\infty} \frac{d x}{(x-a)(x-b)}=\frac{2 i \pi}{a-b}[\Theta(\operatorname{Im} a)-\Theta(\operatorname{Im} b)] \tag{366}
\end{equation*}
$$

A short application is when $a=i$ and $b=-i$, then $a-b=2 i$. Moreover $\Theta(\operatorname{Im} a)=1$ and $\Theta(\operatorname{Im} b)=0$, and so

$$
\begin{equation*}
\int_{-\infty}^{\infty} \frac{d x}{x^{2}+1}=\pi \tag{367}
\end{equation*}
$$

which is correct.
The application important for us is

$$
\begin{align*}
& \int_{-\infty}^{\infty} \frac{d x}{\left(x^{2}-|a|^{2}\right)\left(x^{2}-|b|^{2}\right)}=\int_{-\infty}^{\infty} \frac{d x}{(x-a)\left(x-a^{*}\right)(x-b)\left(x-b^{*}\right)}= \\
& =\frac{1}{a-a^{*}} \frac{1}{b-b^{*}} \int_{-\infty}^{\infty}\left(\frac{1}{x-a}-\frac{1}{x-a^{*}}\right)\left(\frac{1}{x-b}-\frac{1}{x-b^{*}}\right) d x= \\
& =\frac{1}{2 i \operatorname{Im} a} \frac{1}{2 i \operatorname{Im} b} \int_{-\infty}^{\infty}\left(\frac{1}{x-a} \frac{1}{x-b}-\frac{1}{x-a^{*}} \frac{1}{x-b}-\frac{1}{x-a} \frac{1}{x-b^{*}}+\frac{1}{x-a^{*}} \frac{1}{x-b^{*}}\right) \tag{368}
\end{align*}
$$

We can choose the imaginar part of $a$ and $b$ as we like, since in the original expression the sign of the imaginary part does not appear. So we can $\operatorname{fix} \operatorname{Im} a, \operatorname{Im} b>0$. Then we find

$$
\begin{equation*}
\int_{-\infty}^{\infty} \frac{d x}{2 \pi} \frac{1}{\left(x^{2}-|a|^{2}\right)\left(x^{2}-|b|^{2}\right)}=\frac{i}{(2 i \operatorname{Im} a)(2 i \operatorname{Im} b)}\left[\frac{1}{a^{*}-b}-\frac{1}{a-b^{*}}\right]=\frac{1}{2} \frac{\operatorname{Im} a+\operatorname{Im} b}{\left|a-b^{*}\right|^{2} \operatorname{Im} a \operatorname{Im} b} \tag{369}
\end{equation*}
$$

## A. 3 Fierz identities

In the $L_{4} \equiv 4 \times 4$ matrix space we can provide a basis by the 16 gamma matrices

$$
\begin{equation*}
\Gamma^{a}=\left\{1, \gamma_{5}, \gamma_{\mu}, \gamma_{5} \gamma_{\mu}, \sigma_{\mu \nu}\right\} \tag{370}
\end{equation*}
$$

where

$$
\begin{equation*}
\sigma_{\mu \nu}=\frac{i}{2}\left[\gamma_{\mu}, \gamma_{\nu}\right] \tag{371}
\end{equation*}
$$

In the $L^{4} \times L^{4}$ space, therefore, $\Gamma^{a} \times \Gamma^{b}$ provides a basis; with indices it means $\Gamma_{i j}^{a} \Gamma_{i^{\prime} j^{\prime}}^{b}$. Any 4 -index tensor can be expressed through this basis. In particular with an index change we have such a tensor:

$$
\begin{equation*}
\Gamma_{i j^{\prime}}^{a} \Gamma_{i^{\prime} j}^{b}=\sum_{c d} \mathcal{C}^{a b c d} \Gamma_{i j}^{c} \Gamma_{i^{\prime} j^{\prime}}^{d} . \tag{372}
\end{equation*}
$$

The $\mathcal{C}^{a b}$ (Fierz) coefficients can be obtained by taking a trace. If the basis is normalized as

$$
\begin{equation*}
\operatorname{Tr} \Gamma^{a} \Gamma_{b}=d_{\gamma} \delta^{a b} \tag{373}
\end{equation*}
$$

(where the lower indices indicate Lorentz-index pulling down) then we have

$$
\begin{equation*}
\sum_{c d} C^{a b c^{\prime} d^{\prime}}\left(\sum_{i j} \Gamma_{i j}^{c^{\prime}} \Gamma_{c, j i}\right)\left(\sum_{i^{\prime} j^{\prime}} \Gamma_{i^{\prime} j^{\prime} \Gamma^{d^{\prime}}} \Gamma_{d, j^{\prime} i^{\prime}}\right)=d_{\gamma}^{2} \mathcal{C}^{a b c d} \tag{374}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
\mathcal{C}^{a b c d}=\frac{1}{d_{\gamma}^{2}} \sum_{i j i^{\prime} j^{\prime}} \Gamma_{i j^{\prime}}^{a} \Gamma_{i^{\prime} j}^{b} \Gamma_{c, j i} \Gamma_{d, j^{\prime} i^{\prime}}=\frac{1}{d_{\gamma}^{2}} \operatorname{Tr} \Gamma^{a} \Gamma_{d} \Gamma^{b} \Gamma_{c} . \tag{375}
\end{equation*}
$$

As a special application, we have

$$
\begin{equation*}
\sum_{i j i^{\prime} j^{\prime}} \bar{\psi}_{i} \Gamma_{i j^{\prime}}^{a} \eta_{j^{\prime}} \bar{\eta}_{i^{\prime}} \Gamma_{i^{\prime} j}^{b} \psi_{j}=-\sum_{c d} \mathcal{c}^{a b c d} \sum_{i j i^{\prime} j^{\prime}} \bar{\psi}_{i} \Gamma_{i j}^{c} \psi_{j} \bar{\eta}_{i^{\prime}} \Gamma_{i^{\prime} j^{\prime}}^{d} \eta_{j^{\prime}}, \tag{376}
\end{equation*}
$$

which can be written as

$$
\begin{equation*}
\bar{\psi} \Gamma^{a} \eta \bar{\eta} \Gamma^{b} \psi=-\sum_{c d} \mathcal{C}^{a b c d} \bar{\psi} \Gamma^{c} \psi \bar{\eta} \Gamma^{d} \eta . \tag{377}
\end{equation*}
$$

If $\eta=\psi$ this results in the Fierz-identities:

$$
\begin{equation*}
\bar{\psi} \Gamma^{a} \psi \bar{\psi} \Gamma^{b} \psi=-\sum_{c d} \mathcal{C}^{a b c d} \bar{\psi} \Gamma^{c} \psi \bar{\psi} \Gamma^{d} \psi . \tag{378}
\end{equation*}
$$

We need the projection onto the $\Gamma^{1} \equiv 1$ combinations. Then we need

$$
\begin{equation*}
\mathcal{C}^{a b 11}=\frac{1}{d_{\gamma}^{2}} \operatorname{Tr} \Gamma^{a} \Gamma^{b}=\frac{1}{d_{\gamma}} \delta_{b}^{a} g_{b b}, \tag{379}
\end{equation*}
$$

where $g_{b b}= \pm$ depending on whether we have sign change when pulling down the indices. This means that all Lorentz-invariant combination has the same projection onto the unity:

$$
\begin{equation*}
\bar{\psi} \Gamma^{a} \psi \bar{\psi} \Gamma_{a} \psi=-\frac{1}{d_{\gamma}}(\bar{\psi} \psi)^{2}+\ldots \tag{380}
\end{equation*}
$$

## A. 4 Symmetric Fourier transformation

LEt us consider a series $\Phi_{j} j=\{-(N-1) / 2, \ldots(N-1) / 2\}$ where $N$ is an odd number. This series contains $N$ elements. Its Fourier transformation reads

$$
\begin{equation*}
\tilde{\Phi}_{n}=\sum_{j=-(N-1) / 2}^{(N-1) / 2} e^{i 2 \pi \frac{j n}{N}} \Phi_{j} \tag{381}
\end{equation*}
$$

The inverse relation reads

$$
\begin{equation*}
\Phi_{j}=\frac{1}{N} \sum_{n=-(N-1) / 2}^{(N-1) / 2} e^{-i 2 \pi \frac{j n}{N}} \tilde{\Phi}_{n} \tag{382}
\end{equation*}
$$

To prove it we use

$$
\begin{equation*}
\sum_{n=-M}^{M} q^{n}=q^{-M} \sum_{n=0}^{2 M} q^{n}=q^{-M} \frac{q^{2 M+1}-1}{q-1}=\frac{q^{M+1 / 2}-q^{-M-1 / 2}}{q^{1 / 2}-q^{-1 / 2}} \tag{383}
\end{equation*}
$$

therefore

$$
\begin{equation*}
\sum_{n=-(N-1) / 2}^{(N-1) / 2} e^{-i 2 \pi \frac{j n}{N}} \tilde{\Phi}_{n}=\sum_{\ell=-(N-1) / 2}^{(N-1) / 2} \Phi_{\ell} \sum_{n=-(N-1) / 2}^{(N-1) / 2} e^{-i 2 \pi \frac{(j-\ell) n}{N}}=\sum_{\ell=-(N-1) / 2}^{(N-1) / 2} \Phi_{\ell} \frac{e^{-i \pi(j-\ell) n}-e^{i \pi(j-\ell) n}}{e^{-i \pi \frac{j-\ell}{N}}-e^{-i \pi \frac{j-\ell}{N}}} . \tag{384}
\end{equation*}
$$

The last expression is 0 if $j \neq \ell$ and $N$ if $j=\ell$, this proves the inverse relation.
If $N \rightarrow \infty$ we introduce $p=2 \pi n / N$, then the Fourier trasformation reads

$$
\begin{equation*}
\tilde{\Phi}(p)=\sum_{j=-\infty}^{\infty} e^{i j p} \Phi_{j} \tag{385}
\end{equation*}
$$

while the inverse relation, using $d p /(2 \pi)=1 / N$, and $p_{\max }=\pi(N-1) / N \rightarrow \pi$, reads

$$
\begin{equation*}
\Phi_{j}=\int_{-\pi}^{\pi} \frac{d p}{2 \pi} e^{-i j p} \tilde{\Phi}(p) \tag{386}
\end{equation*}
$$

In particular if $\tilde{\Phi}(p)=2 \pi \delta(p)$ then $\Phi_{j}=1$, which means

$$
\begin{equation*}
2 \pi \delta(p)=\sum_{j=-\infty}^{\infty} e^{i j p} \tag{387}
\end{equation*}
$$

We can generalize it with a "lattice spacing". Then we introduce $p=2 \pi n /(N a)$, where $a$ is an arbitrary number and we introduce also $\varphi_{j}=\Phi_{j} / a$. Then the Fourier trasformation reads

$$
\begin{equation*}
\tilde{\Phi}(p)=a \sum_{j=-\infty}^{\infty} e^{i j a p} \varphi_{j} \tag{388}
\end{equation*}
$$

The inverse relation, using $a d p /(2 \pi)=1 / N$, and $p_{\max }=\pi(N-1) /(N a) \rightarrow \pi / a$, reads

$$
\begin{equation*}
\varphi_{j}=\int_{-\pi / a}^{\pi / a} \frac{d p}{2 \pi} e^{-i j a p} \tilde{\Phi}(p) \tag{389}
\end{equation*}
$$

In particular if $\tilde{\Phi}(p)=2 \pi \delta(p)$ then $\varphi_{j}=1$, which means

$$
\begin{equation*}
2 \pi \delta(p)=a \sum_{j=-\infty}^{\infty} e^{i j a p} \tag{390}
\end{equation*}
$$

From here the continuum limit can be obtained by $a \rightarrow 0$ limit, where $x=j a$ and

$$
\begin{equation*}
\tilde{\Phi}(p)=\int_{-\infty}^{\infty} d x e^{i x p} \varphi(x), \quad \varphi(x)=\int_{-\infty}^{\infty} \frac{d p}{2 \pi} e^{-i x p} \tilde{\Phi}(p) \tag{391}
\end{equation*}
$$

and

$$
\begin{equation*}
2 \pi \delta(p)=\int_{-\infty}^{\infty} d x e^{i x p} \tag{392}
\end{equation*}
$$


[^0]:    ${ }^{1}$ To be correct to have a comprehensive description a lot of advanced tricks have to be applied first of all in the low energy regime. We mention some of these methods in this note.

[^1]:    ${ }^{2}$ A power set is usually defined as a set of all subsets of the basic set, but it is nothing then the function $f: A \rightarrow\{0,1\}$, and thus the number of elements is $2^{D} A$.

[^2]:    ${ }^{3}$ Besides this this regulator is often optimal, meaning that the result changes least when one chooses this regulator.

[^3]:    ${ }^{4}$ At higher dimensions there is also divergent behaviour, but there the important point is that $m_{k}^{2} \sim-k^{2}$ is reached, so we must treat it in the broken symmetry phase.

[^4]:    ${ }^{5}$ M. Hasenbusch, K. Pinn and S. Vinti, Phys.Rev. B59 (1999) 11471-11483, hep-lat/9806012

